Preorder-Constrained Simulation

(Early Idea)

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Outline

- Overview
- Preorder-Constrained Simulation without up-to
- Preorder-Constrained Simulation with up-to
- Conclusion and Future Work

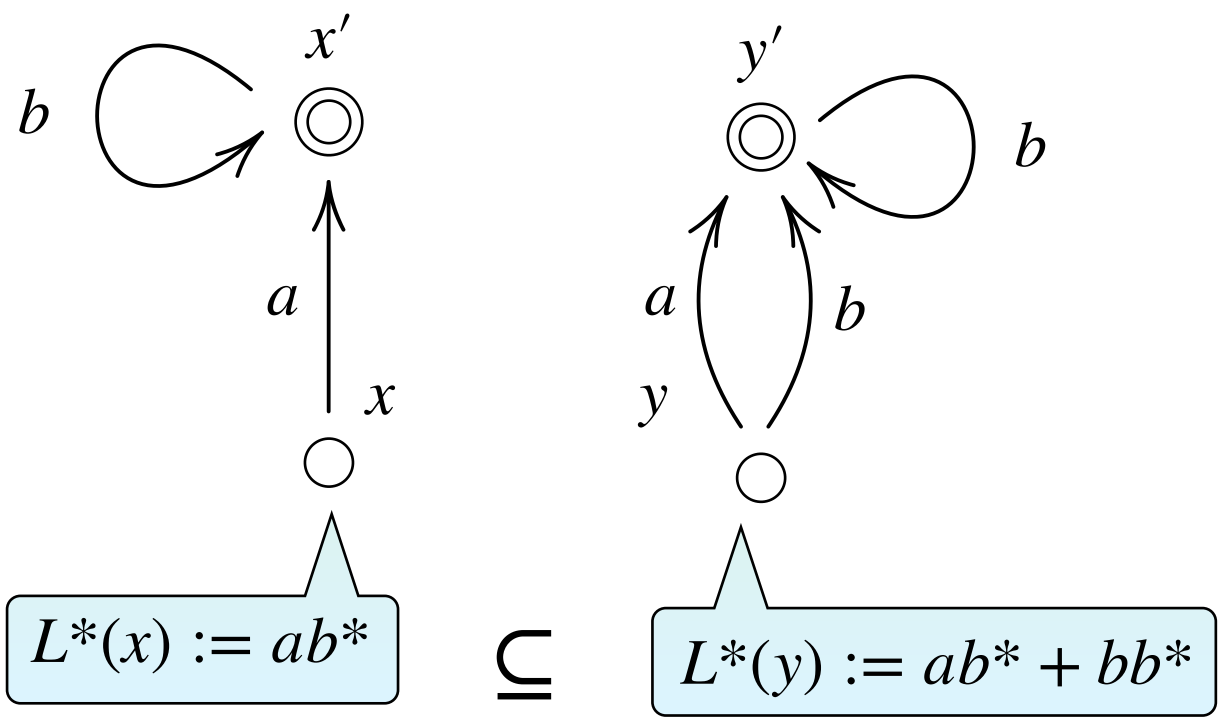
Simulation

for behavioral **equivalence:**
bisimulation

- Step-wise formalization for behavioral inclusion

- Useful for proving trace inclusion

- Example:



$L^*(x) := ab^*$ $\subseteq$ $L^*(y) := ab^* + bb^*$

PSPACE-complete

trace inclusion (language inclusion)

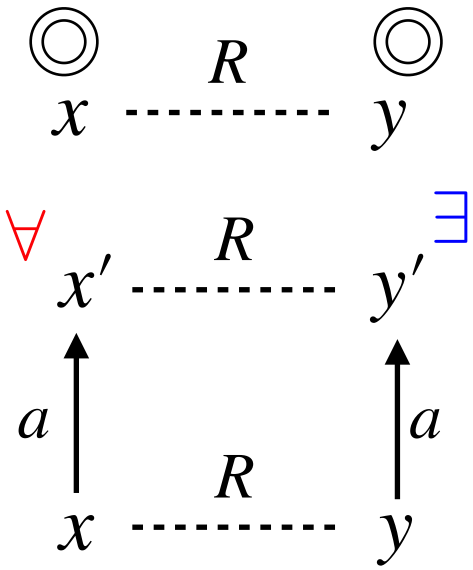
Forward Simulation [Lynch & Vaandrager, ’95]

Definition:

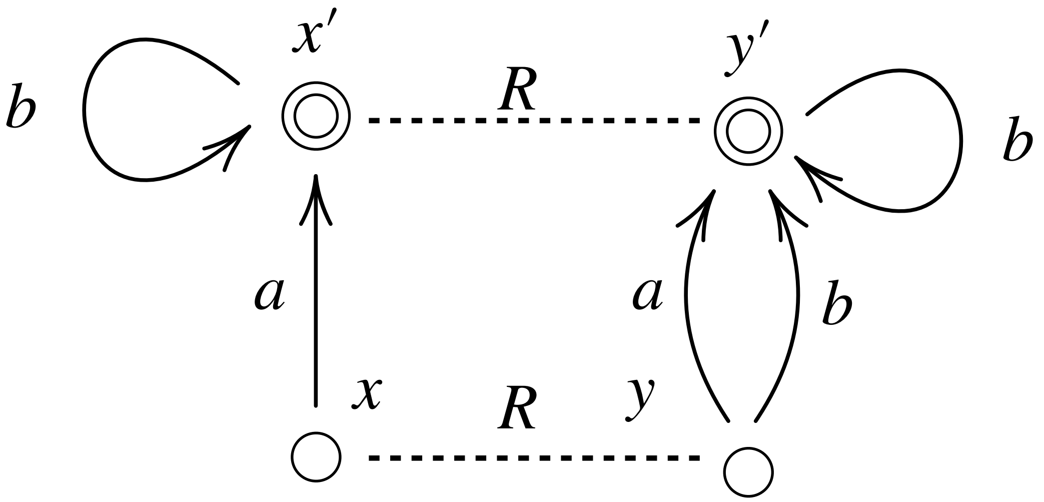
A ***forward simulation*** from $(c : X \to \mathscr{P}(\Sigma \times X), F_1 \subseteq X)$ to $(d : Y \to \mathscr{P}(\Sigma \times Y), F_2 \subseteq Y)$ is a relation $R \subseteq X \times Y$ such that

$\forall (x, y) \in R\,.$

- $x \in F_1 \implies y \in F_2$ and
- $\forall (a, x') \in c(x)\,.\ \exists y'$ s.t. $(a, y') \in d(y)\,.\ x' R y'$



in P



$L^*(x) := ab^*$ $\subseteq$ $L^*(y) := ab^* + bb^*$

trace inclusion

Theorem (**soundness**):

If R is a forward simulation,

$$xRy \implies L^*(x) \subseteq L^*(y)$$

PSPACE-complete

Various Simulation Notions

- Simulation up-to
- Weak Simulation
- Improvement
- Preorder-Constrained Simulation

Simulation up-to & Weak Simulation

Simulation up-to

- Enhancement with prior knowledge on trace inclusion

$L^*(x') \subseteq L^*(x'')$ $\quad$ $L^*(y') \subseteq L^*(y'')$

$$\begin{tikzcd}
x' \arrow[r, dash, dashed] & x'' \arrow[r, dash, dashed, "R"] & y'' \arrow[r, dash, dashed] & y' \\
x \arrow[u, "a"] \arrow[rrr, dash, dashed, "R"] & & & y \arrow[u, "a"']
\end{tikzcd}
\quad \Longrightarrow \quad L^*(x) \subseteq L^*(y)$$

Weak simulation

- Simulation for systems with **silent moves,** e.g. $c : X \to \mathscr{P}((\{\tau\} + \Sigma) \times X)$

silent letter

$$\begin{tikzcd}
x' \arrow[r, dash, dashed, "R"] & y' \\
x \arrow[u, "\tau"] \arrow[r, dash, dashed, "R"] & y \arrow[u, dotted, "\tau^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
x' \arrow[r, dash, dashed, "R"] & y' \\
 & {} \arrow[u, dotted, "\tau^*"'] \\
 & {} \arrow[u, "a"'] \\
x \arrow[uuu, "a"] \arrow[r, dash, dashed, "R"] & y \arrow[u, dotted, "\tau^*"']
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{array}{l} \mathrm{remove}_\tau(L^*(x)) \\ \quad \subseteq \mathrm{remove}_\tau(L^*(y)) \end{array}$$

weak trace inclusion

Naive combination of weak and up-to is NOT sound

→ need special care, e.g. [Pous, ’05]

Simulations in Program Semantics Literature

Improvement [Accattoli, Dal Lago & Vanoni, 2020]

$e \to^{n} w \not\to$
$\implies \exists n' : n \geq n' .\ e' \to^{n'} w' \not\to$

- (Bi)simulation for comparing **reduction lengths** of λ-terms
- Automata-theoretically: for deterministic & unlabeled systems $(c : X \to X, F_1 \subseteq X)$

$\cong c : X \to \{\tau\} \times X$

variant of weak simulation

- Simulate multiple steps with multiple steps

$$\begin{tikzcd} x' \arrow[r, dash, dashed, "R"] & y' \\ x \arrow[u, dotted, "\exists\, m > 0 \text{ steps}"] \arrow[r, dash, dashed, "R"'] \arrow[ur, phantom, "m \geq n"] & y \arrow[u, dotted, "n \text{ steps}\, \exists"'] \end{tikzcd}$$

$e \to^{n} w \not\to$
$\implies \exists n' : nQn' .\ e' \to^{n'} w' \not\to$

Preorder-constrained simulation [Muroya, PhD thesis, 2020]

$\cong$ (One-directional) improvement + up-to + generalization

special care required

(+ some restriction)

- Parameterized by a preorder $Q \subseteq \mathbb{N} \times \mathbb{N}$ closed under addition

(i.e. $iQi' \land jQj' \implies (i+j)Q(i'+j')$)

$$\begin{tikzcd} x' \arrow[r, dash, dashed] & x'' \arrow[r, dash, dashed, "R"] & y'' \arrow[r, dash, dashed] & y' \\ x \arrow[u, dotted, "m > 0"] \arrow[rrr, dash, dashed, "R"'] \arrow[urrr, phantom, "mQn"] & & & y \arrow[u, dotted, "n"'] \end{tikzcd}$$

Main Contribution

Preorder-constrained simulation

for

nondeterministic automata

- Parameterized by a preorder $Q \subseteq \mathbb{N} \times \mathbb{N}$ closed under addition (i.e. $iQi' \wedge jQj' \implies (i+j)Q(i'+j')$)

$$\begin{tikzcd}
x' \arrow[r, dash, dashed, "R"] & y' \\
x \arrow[u, dotted, "m>0"] \arrow[r, dash, dashed, "R"'] \arrow[ur, phantom, "mQn"] & y \arrow[u, dotted, "n"']
\end{tikzcd}$$

- Parameterized by a preorder $\mathcal{Q} \subseteq \Sigma^* \times \Sigma^*$ closed under concatenation (i.e. $w\mathcal{Q}w' \wedge v\mathcal{Q}v' \implies wv\mathcal{Q}w'v'$)

$$\begin{tikzcd}
x' \arrow[r, dash, dashed, "R"] & y' \\
x \arrow[u, dotted, "w"] \arrow[r, dash, dashed, "R"'] \arrow[ur, phantom, "w\mathcal{Q}w'"] & y \arrow[u, dotted, "w'"']
\end{tikzcd}$$

Theorem (**soundness**):

$$xRy \implies \forall w \in L^*(x) \,.\, \exists w' \in L^*(y) \,.\, w\mathcal{Q}w'$$

$\mathcal{Q}$**-trace inclusion** $L^*(x) \preceq_{\mathcal{Q}} L^*(y)$

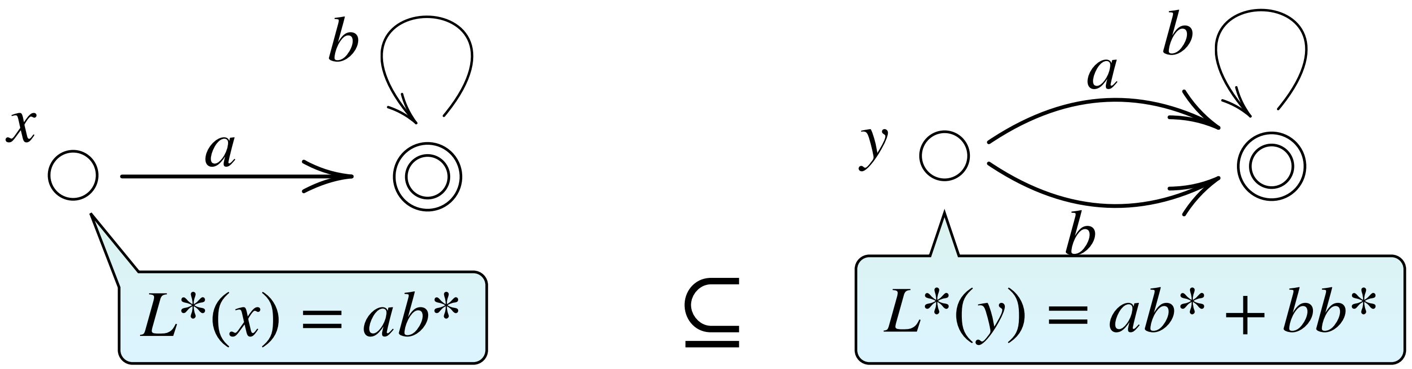
Examples for $\mathcal{Q}$-trace Inclusion

$$\mathcal{Q} \subseteq \Sigma^* \times \Sigma^*$$

$$\forall w \in L^*(x)\,.\,\exists w' \in L^*(y)\,.\, w\mathcal{Q}w'$$

- When $w\mathcal{Q}w' \overset{\text{def}}{\iff} w = w'$

  $\mathcal{Q}$-trace inclusion $\iff L^*(x) \subseteq L^*(y)$ **(finite trace inclusion)**

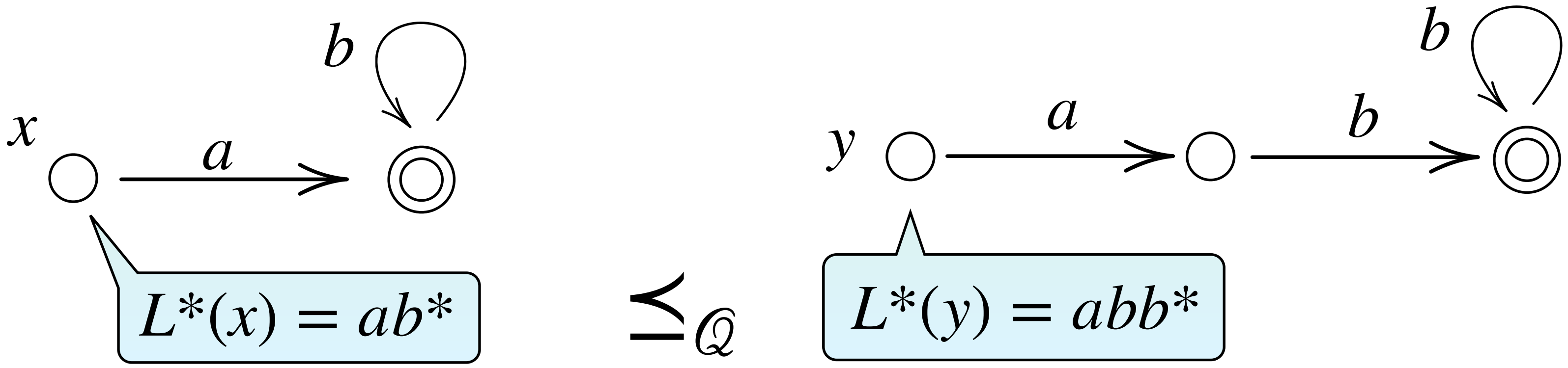


$L^*(x) = ab^*$ $\subseteq$ $L^*(y) = ab^* + bb^*$

- When $w\mathcal{Q}w' \overset{\text{def}}{\iff} w$ is a substring of w'

  $\mathcal{Q}$-trace inclusion $\iff \forall w \in L^*(x)\,.\,\exists w' \in L^*(y)\,.\, w$ is a substring of w'

  (trace inclusion wrt. substring)



$L^*(x) = ab^*$ $\preceq_{\mathcal{Q}}$ $L^*(y) = abb^*$

Examples for $\mathcal{Q}$-trace Inclusion

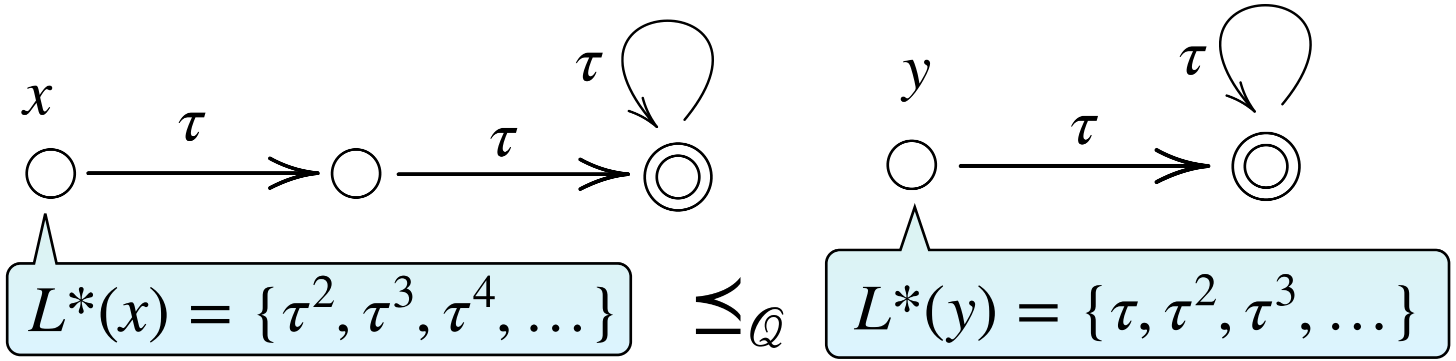
$\mathcal{Q} \subseteq \Sigma^* \times \Sigma^*$

$\forall w \in L^*(x) \,.\, \exists w' \in L^*(y) \,.\, w\mathcal{Q}w'$

- When $\Sigma = \{\tau\}$ and $w\mathcal{Q}w' \overset{\text{def}}{\iff} |w| \geq |w'|$

$$\mathcal{Q}\text{-trace inclusion} \iff \min \mathsf{length}(x \to \cdots \to \checkmark) \geq \min \mathsf{length}(y \to \cdots \to \checkmark)$$

(compare minimum distance to accepting state)

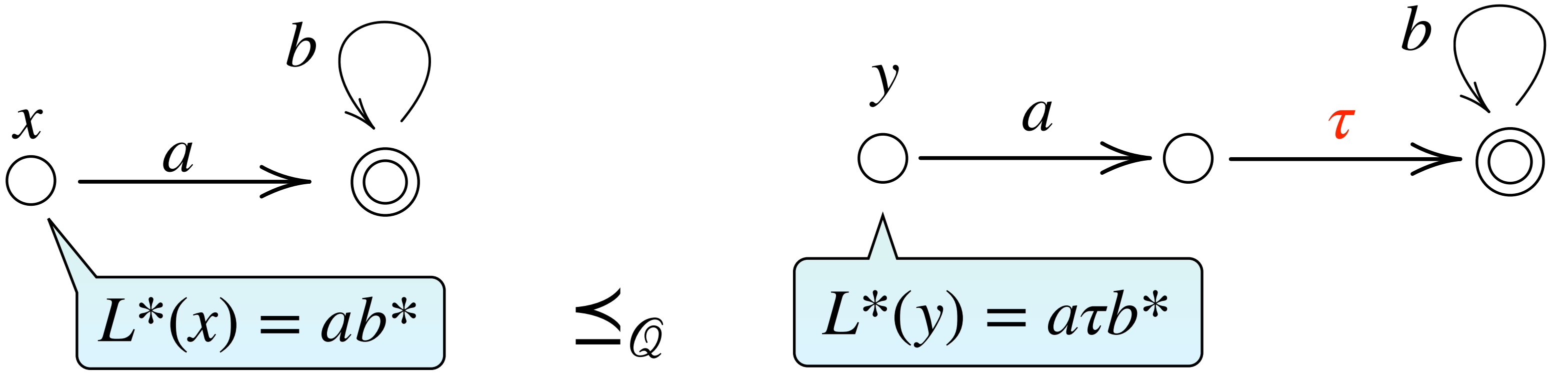


$L^*(x) = \{\tau^2, \tau^3, \tau^4, \ldots\} \preceq_{\mathcal{Q}} L^*(y) = \{\tau, \tau^2, \tau^3, \ldots\}$

- When $\Sigma = \{\tau\} + \Sigma'$ and $w\mathcal{Q}w' \overset{\text{def}}{\iff} \mathsf{remove}_\tau(w) = \mathsf{remove}_\tau(w')$

$$\mathcal{Q}\text{-trace inclusion} \iff \mathsf{remove}_\tau(L^*(x)) \subseteq \mathsf{remove}_\tau(L^*(y))$$

(weak trace inclusion)



$L^*(x) = ab^* \preceq_{\mathcal{Q}} L^*(y) = a\tau b^*$

- And more (e.g. weighted automata)

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Towards Generalization

preorder-constrained simulation
for **deterministic & unlabeled systems**
[Accattoli, Dal Lago & Vanoni, 2020] [Muroya, Phd thesis]

$$\begin{tikzcd}
x' \arrow[r, dash, dashed, "R"] & y' \\
x \arrow[u, dotted, "\exists\ m > 0"] \arrow[r, dash, dashed, "R"'] \arrow[ur, phantom, "mQn"] & y \arrow[u, dotted, "n\ \exists"']
\end{tikzcd}$$

+

forward simulation
for **nondeterministic automata**
[Lynch & Vaandrager, ‘95]

$$\begin{tikzcd}
\forall\ x' \arrow[r, dash, dashed, "R"] & y'\ \exists \\
x \arrow[u, "a"] \arrow[r, dash, dashed, "R"'] & y \arrow[u, "a"']
\end{tikzcd}$$

preorder-constrained simulation
for **nondeterministic automata**

$$\begin{tikzcd}
x' \arrow[r, dash, dashed, "R"] & y' \\
x \arrow[u, dotted, "w \neq \varepsilon"] \arrow[r, dash, dashed, "R"'] \arrow[ur, phantom, "w\mathcal{Q}w'"] & y \arrow[u, dotted, "w'"']
\end{tikzcd}$$

w and x' are quantified by ∀
$|w|$ is quantified by ∃

Main Result:
Preorder-Constrained Simulation for Nondeterministic Automata

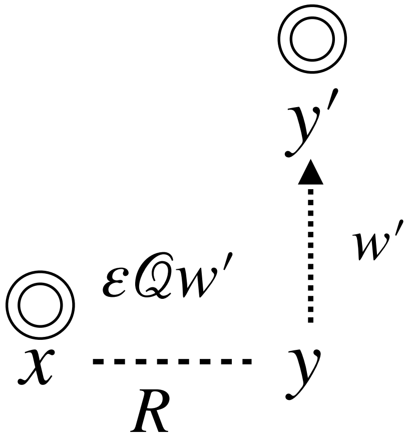
Definition:

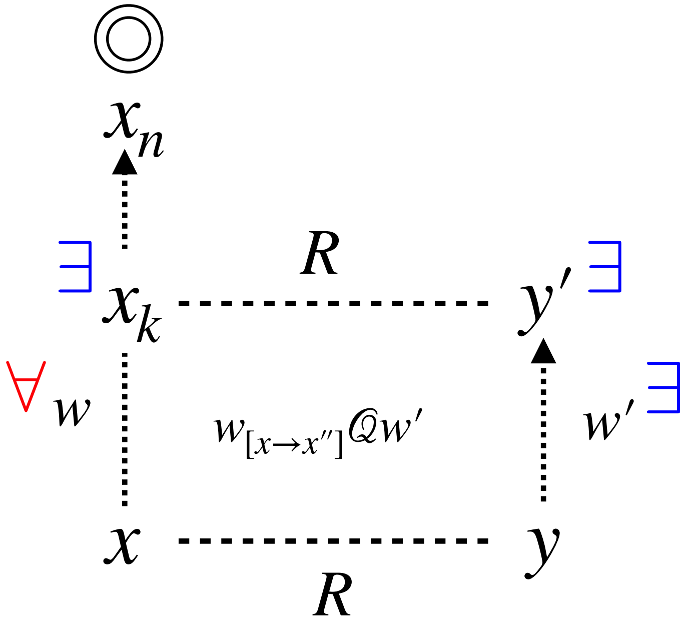
Let $\mathcal{Q} \subseteq \Sigma^* \times \Sigma^*$ be a preorder. A ***$\mathcal{Q}$-constrained simulation*** from $(c : X \to \mathscr{P}(\Sigma \times X), F_1 \subseteq X)$ to $(d : Y \to \mathscr{P}(\Sigma \times Y), F_2 \subseteq Y)$ is $R \subseteq X \times Y$ s.t.

$\forall (x, y) \in R\,.$

- $x \in F_1 \implies \exists w' \in \Sigma^*\,.\ \varepsilon \mathcal{Q} w', y \overset{w'^*}{\to} y' \in F_2$
- $\forall a_1 \ldots a_n \in \Sigma^*\,.\ \forall x_1 \ldots x_n \in X_1^*\,.\ x \overset{a_1}{\to} x_1 \overset{a_2}{\to} \cdots \overset{a_n}{\to} x_n \in F_1$

  $\implies \exists k \in \{1, \ldots, n\}\,.\ \exists w' \in \Sigma^*\,.\ a_1 \ldots a_k \mathcal{Q} w', y \overset{w'^*}{\to} y'$ and $x_k R y'$





Theorem (**soundness**):

When $\mathcal{Q}$ is closed under concatenation,

$$xRy \implies \forall w \in L^*(x)\,.\ \exists w' \in L^*(y)\,.\ w \mathcal{Q} w'$$

Characterization as Safety Game

$S_\forall := \{\checkmark\} + \Sigma^* \times X \times Y$ **(state space for Challenger)**

queue

$S_\exists := \{\text{last\_turn}\} \times \Sigma^* \times X \times Y + \Sigma^+ \times X \times Y$ **(state space for Simulator)**

$\rightarrow_\forall \subseteq S_\forall \times S_\exists$ is given by:

$$\left\{\left((w,x,y),(wa,x',y)\right) \mid x \xrightarrow{a} x'\right\} \cup \left\{\left((w,x,y),(\text{last\_turn},w,x,y)\right) \mid x \in F_1\right\}$$

choose successor state and enqueue **declare "last turn" (possible when x is accepting)**

$\rightarrow_\exists \subseteq S_\exists \times S_\forall$ is given by:

$$\left\{\left((w,x',y),(w,x',y)\right)\right\} \cup \left\{\left((w,x',y),(\varepsilon,x',y')\right) \mid y \xrightarrow{w'}{}^{*} y', w\mathcal{Q}w'\right\}$$

pass the turn **dequeue all, and move**

$$\cup \left\{\left((\text{last\_turn},w,x,y),\checkmark\right) \mid y \xrightarrow{w'}{}^{*} y' \in F_2, w\mathcal{Q}w'\right\}$$

reach accepting state, and win the game

- **Simulator loses iff it gets stuck (i.e. infinite play is winning)**

Conjecture:

Simulator is winning from (ε, x, y) $\iff$

$(x,y) \in R$ *for some* $\mathcal{Q}$*-constrained simulation*

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Preorder-Constrained Simulation with up-to

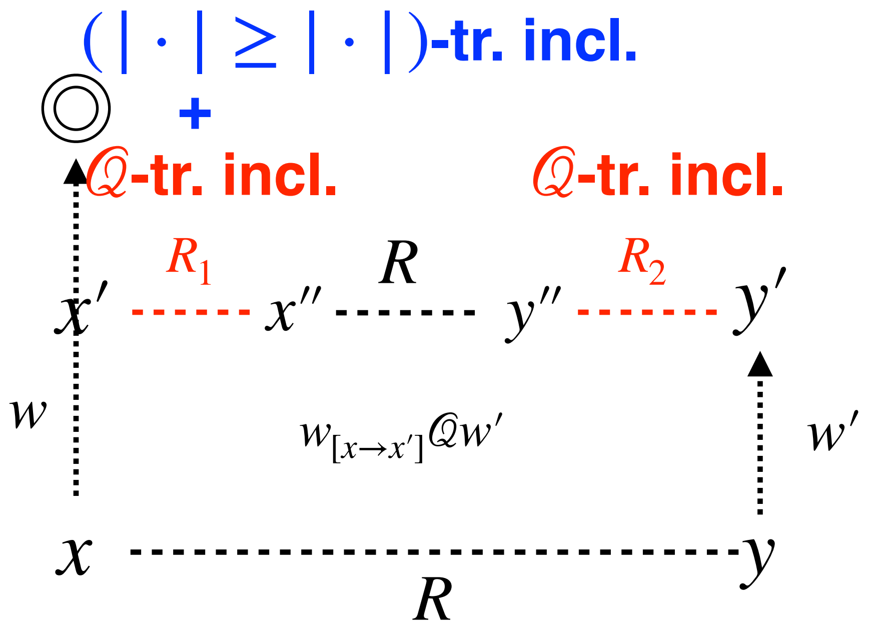
Definition:

Let $\mathcal{Q} \subseteq \Sigma^* \times \Sigma^*$. A ***$\mathcal{Q}$-constrained simulation* up-to** $(R_1 \subseteq X \times X, R_2 \subseteq Y \times Y)$ from $(c : X \to \mathscr{P}(\Sigma \times X), F_1 \subseteq X)$ to $(d : Y \to \mathscr{P}(\Sigma \times Y), F_2 \subseteq Y)$ is $R \subseteq X \times Y$ s.t.

$$\forall (x, y) \in R\,. \quad \text{- } x \in F_1 \implies \exists w' \in \Sigma^*\,.\, \varepsilon \mathcal{Q} w', y \xrightarrow{w'}{}^{*} y' \in F_2$$

$$\text{- } \forall a_1 \ldots a_n \in \Sigma^*\,.\, \forall x_1 \ldots x_n \in X_1^*\,.\, x \xrightarrow{a_1} x_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} x_n \in F_1$$

$$\implies \exists k \in \{1, \ldots, n\}\,.\, \exists w' \in \Sigma^*\,.\, a_1 \ldots a_k \mathcal{Q} w', y \xrightarrow{w'}{}^{*} y' \text{ and } x_k R_1 R R_2 y'$$



Theorem (**soundness**):

When $\mathcal{Q}$ is closed under concatenation,

xR_1x' and yR_2y' imply $\mathcal{Q}$-trace inclusion, and

xR_1x' implies $(|\cdot| \geq |\cdot|)$-trace inclusion,

$$xRy \implies$$

$$\forall w \in L^*(x)\,.\, \exists w' \in L^*(y)\,.\, w \mathcal{Q} w'$$

for safely combining weak & up-to

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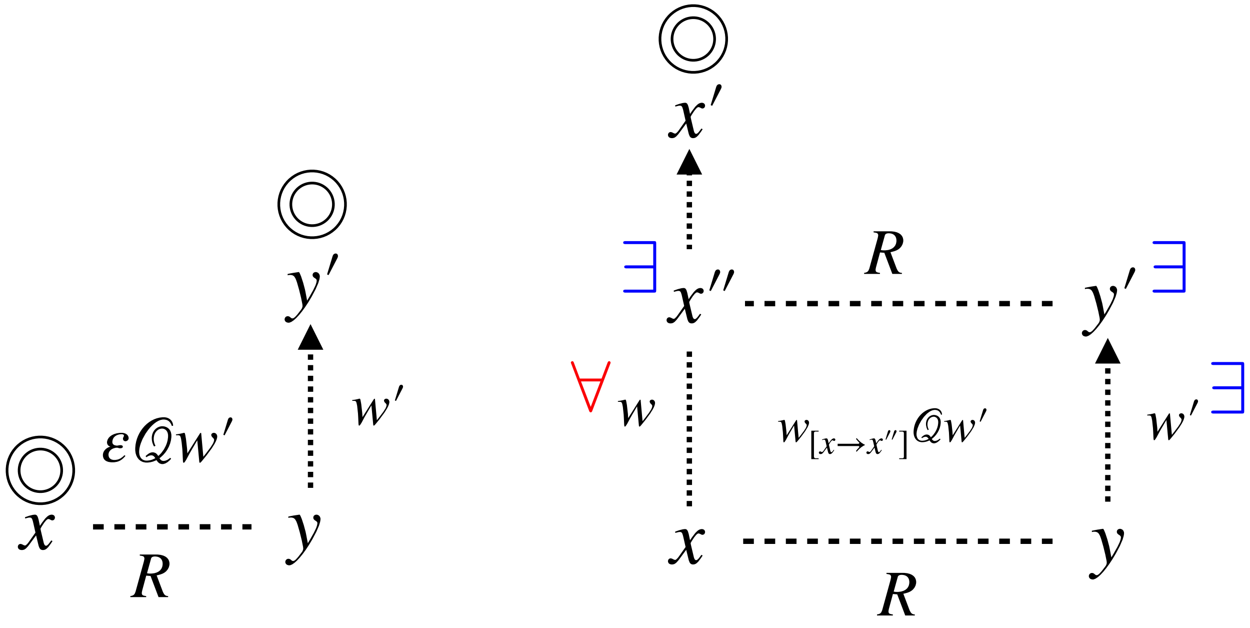
Conclusion

- New simulation notion for witnessing **$\mathcal{Q}$-trace inclusion**

$$\forall w \in L^*(x)\,.\,\exists w' \in L^*(y)\,.\,w \mathcal{Q} w'$$

- Enhancement with up-to



Future Directions

- Extension to **infinitary** trace inclusion

- Relaxing condition in up-to

Theorem (**soundness**):

When $\mathcal{Q}$ is closed under concatenation,

xR_1x' and yR_2y' imply $\mathcal{Q}$-trace inclusion, and

xR_1x' implies $(|\cdot| \geq |\cdot|)$-trace inclusion,

$$xRy \implies \forall w \in L^*(x)\,.\,\exists w' \in L^*(y)\,.\,w\mathcal{Q}w'$$

Conjecture:
finitely many violation is ok

- Coalgebraic characterization
  - Coalgebraic simulation: [Hughes & Jacobs, '04] [Hasuo, '06]
  - Coalgebraic simulation with queues: [U. & Hasuo, '14]
  - Coalgebraic bisimulation up-to: [Rot, Bonsangue & Rutten, '13]

Addenda

Addendum I: Buffered Simulation

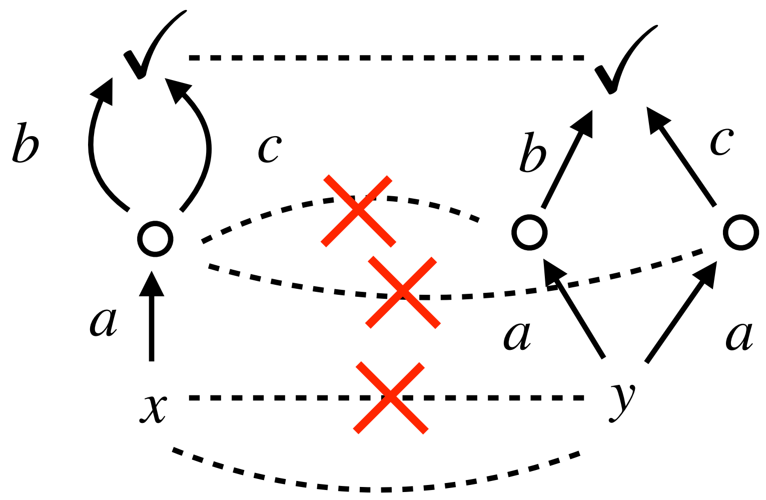
- Simulation with queueing (**buffering**) is used for fixing **incompleteness**
  - e.g. [Hutagalung, Lange & Lozes, AFL 2014] for Büchi automata

Theorem (**soundness**):

If R is a forward simulation,

$$xRy \;\begin{array}{c}\Longrightarrow\\ \cancel{\Longleftarrow}\end{array}\; L^*(x) \subseteq L^*(y)$$

incomplete



- $L^*(x) = L^*(y) = \{ab, ac\}$, but no forward simulation can prove it
- However, it does exist if buffering is allowed

- Preorder-constrained simulation also has this property

Addendum II: Categorical Buffered Simulation

- Kleisli Simulation [Hasuo, ’06]
  - System as a coalgebra in Kleisli category

    $c : X \to FX$ in $\mathscr{K}\ell(T)$ whose homsets are order-enriched

  - Simulation as an oplax homomorphism

$$\begin{tikzcd} FX & FY \arrow[l, "\overline{F}f"'] \\ X \arrow[u, "c"] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[l, "f"] \arrow[u, "d"'] \end{tikzcd}$$

in $\mathscr{K}\ell(T)$

$\overline{F} : \mathscr{K}\ell(T) \to \mathscr{K}\ell(T)$: lifting of F

- Forward partial execution [U. & Hasuo, ’14]

$$\begin{tikzcd} FX \\ X \arrow[u, "c"] \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} F^{n+1}X \\ F^nX \arrow[u, "\overline{F}^n c"] \end{tikzcd}$$

in $\mathscr{K}\ell(T)$

➡

$$\begin{tikzcd} F^{n+1}X & \cdots \arrow[l, "\overline{F}^n c"'] & FX \arrow[l, "\overline{F}c"'] & FY \arrow[l, "\overline{F}f"'] \\ F^nX \arrow[u, "\overline{F}^n c"] \arrow[urr, phantom, "="] & \cdots \arrow[l, "\overline{F}^{n-1}c"] & X \arrow[l, "c"] \arrow[u, "c"] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[l, "f"] \arrow[u, "d"'] \end{tikzcd}$$

in $\mathscr{K}\ell(T)$

  - Essentially the same as buffering one step

e.g. when $F = 1 + \Sigma \times (\,\cdot\,)$, $F^nX = \bigcup_{i \leq n} \Sigma^i \times X$

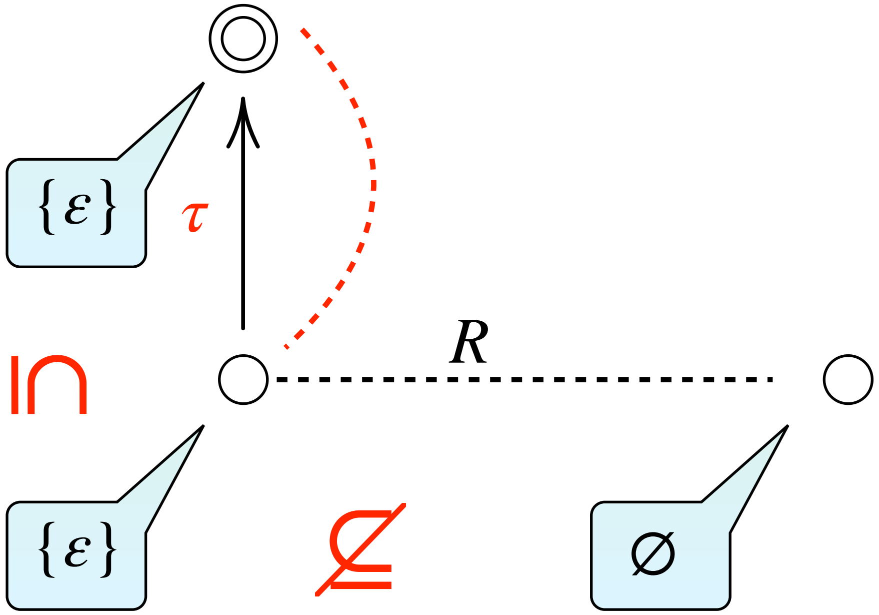
Addendum III: Unsound Weak Simulation up-to

[Sangiorgi & Milner, ’92]

$L^*(x') \subseteq L^*(x'')$ $\quad$ $L^*(y') \subseteq L^*(y'')$

$$\begin{tikzcd} x' \arrow[r, dash, dashed] & x'' \arrow[r, dash, dashed, "R"] & y'' \arrow[r, dash, dashed] & y' \\ x \arrow[u, "a"] \arrow[rrr, dash, dashed, "R"] & & & y \arrow[u, "a"'] \end{tikzcd}$$

$$\begin{tikzcd} x' \arrow[r, dash, dashed, "R"] & y' \\ x \arrow[u, "\tau"] \arrow[r, dash, dashed, "R"] & y \arrow[u, dotted, "\tau^*"'] \end{tikzcd}$$



Addendum IV: Towards Computation

- Conjecture: preorder-constrained simulation is not only sound but also **complete**

Conjecture (**completeness**):

When $\mathcal{Q}$ is closed under concatenation,

$$xRy \Longleftarrow \quad \forall w \in L^*(x)\,.\,\exists w' \in L^*(y)\,.\, w\mathcal{Q}w'$$

Hard to compute

- We may have to finitely restrict the size of queue

Addendum IV: Towards Computation

$$S_\forall := \{\checkmark\} + \bigcup_{i\le M} \Sigma^i \times X \times Y$$

queue of size M

$$S_\exists := \{\text{last\_turn}\} \times \bigcup_{i\le M} \Sigma^i \times X \times Y + \bigcup_{1<i\le M} \Sigma^i \times X \times Y$$

$\to_\forall \subseteq S_\forall \times S_\exists$ is given by:

$$\Big\{\big((w,x,y),(wa,x',y)\big) \mid x \xrightarrow{a} x'\Big\} \cup \Big\{\big((w,x,y),(\text{last\_turn},w,x,y)\big) \mid x \in F_1\Big\}$$

$\to_\exists \subseteq S_\exists \times S_\forall$ is given by:

$$\Big\{\big((w,x',y),(w,x',y)\big) \mid |w| < M\Big\} \cup \Big\{\big((w,x',y),(\varepsilon,x',y')\big) \mid y \xrightarrow{w'}{}^{*} y', w \mathcal{Q} w'\Big\}$$

pass the turn when queue is not full

$$\cup \Big\{\big((\text{last\_turn},w,x,y),\checkmark\big) \mid y \xrightarrow{w'}{}^{*} y' \in F_2, w \mathcal{Q} w'\Big\}$$

- When M is fixed, solvable in polynomial time
- Bigger M → more simulations & higher time complexity

Addendum V: Examples for $\mathcal{Q}$-trace Inclusion

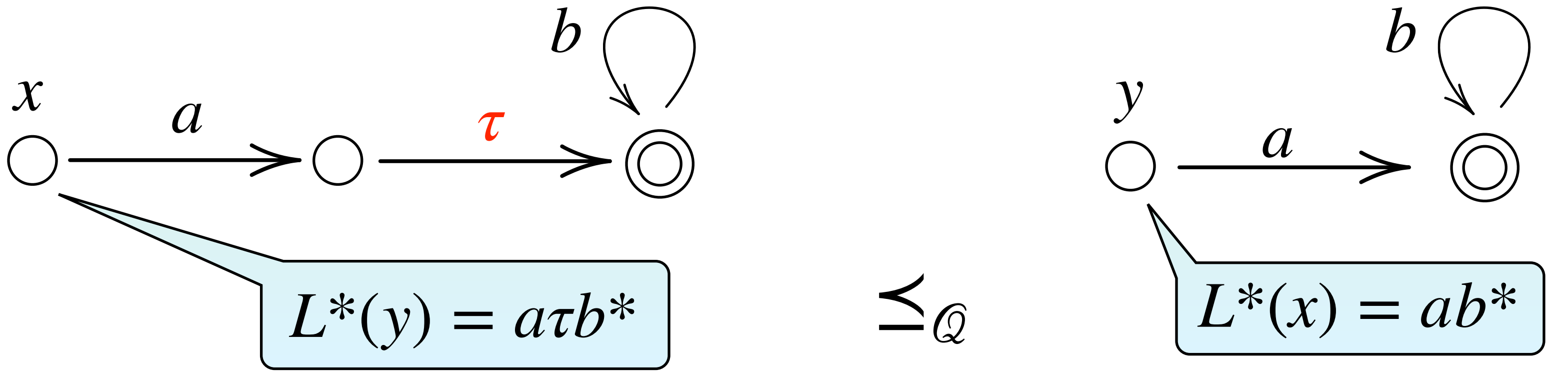
$\mathcal{Q} \subseteq \Sigma^* \times \Sigma^*$

$\forall w \in L^*(x) \,.\, \exists w' \in L^*(y) \,.\, w \mathcal{Q} w'$

- When $\Sigma = \{\tau\} + \Sigma'$ and $w \mathcal{Q} w' \overset{\text{def}}{\iff}$ $\mathrm{remove}_\tau(w) = \mathrm{remove}_\tau(w')$ and $|w| \geq |w'|$

$\mathcal{Q}$-trace inclusion $\iff \forall w \in L^*(x) \,.\, \exists w' \in L^*(y) \,.\, \mathrm{remove}_\tau(w) = \mathrm{remove}_\tau(w')$ and $|w| \geq |w'|$

(weak trace inclusion & compare distance to accepting state)

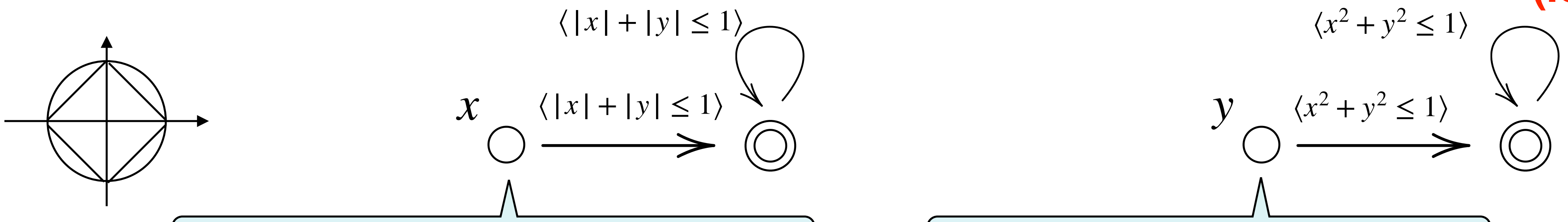


$L^*(y) = a\tau b^*$ $\preceq_{\mathcal{Q}}$ $L^*(x) = ab^*$

- When $\Sigma = \mathscr{P}\mathbb{R}^m$ and $a_1 \ldots a_k \mathcal{Q} a'_1 \ldots a'_{k'} \overset{\text{def}}{\iff}$ $k = k'$ and $\forall i \,.\, a_i \subseteq a'_i$

$\mathcal{Q}$-trace inclusion $\iff \forall a_1 \ldots a_k \in L^*(x) \,.\, \exists a'_1 \ldots a'_k \in L^*(y) \,.\, \forall i \,.\, a_i \subseteq a'_i$

(letter-wise inclusion)


$\langle |x| + |y| \leq 1 \rangle$
x
$\langle |x| + |y| \leq 1 \rangle$
$\langle x^2 + y^2 \leq 1 \rangle$
y
$\langle x^2 + y^2 \leq 1 \rangle$

$L^*(x) = \langle |x| + |y| \leq 1 \rangle \langle |x| + |y| \leq 1 \rangle^*$ $\preceq_{\mathcal{Q}}$ $L^*(x) = \langle x^2 + y^2 \leq 1 \rangle \langle x^2 + y^2 \leq 1 \rangle^*$

Addendum V: Examples for $\mathcal{Q}$-trace Inclusion

$$\mathcal{Q} \subseteq \Sigma^* \times \Sigma^*$$

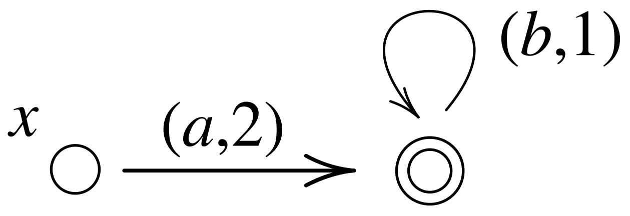
$$\forall w \in L^*(x) \,.\, \exists w' \in L^*(y) \,.\, w \mathcal{Q} w'$$

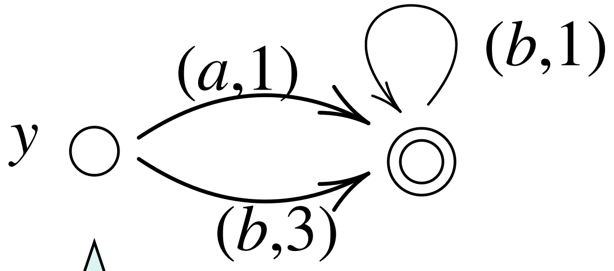
- When $\Sigma = \Sigma' \times \mathbb{N}$ and

$$(a_1, n_1)\dots(a_k, n_k)\mathcal{Q}(a'_1, n'_1)\dots(a'_{k'}, n'_{k'}) \overset{\text{def}}{\iff} k = k', a_1\dots a_k = a'_1\dots a'_{k'} \text{ and } \textstyle\sum_i n_i \geq \sum_j n'_j$$

$$\mathcal{Q}\text{-trace inclusion} \iff \forall a_1\dots a_k \in \Sigma'^* \,.\, \min_{x \xrightarrow{a_1,n_1} \cdots \xrightarrow{a_k,n_k} \checkmark} \textstyle\sum_i n_i \geq \min_{y \xrightarrow{a'_1,n_1} \cdots \xrightarrow{a'_k,n_k} \checkmark} \textstyle\sum_i n'_i$$

(quantitative language inclusion)





$L^*(x) = (a,2)(b,1)^*$ $\preceq_{\mathcal{Q}}$ $L^*(y) = (a,1)(b,1)^* + (b,3)(b,1)^*$

$a \mapsto 2$
$ab \mapsto 3$
$abb \mapsto 4$
$\vdots$

$a \mapsto 1$
$ab \mapsto 2$
$abb \mapsto 3$
$\vdots$

$b \mapsto 3$
$bb \mapsto 4$
$bbb \mapsto 5$
$\vdots$

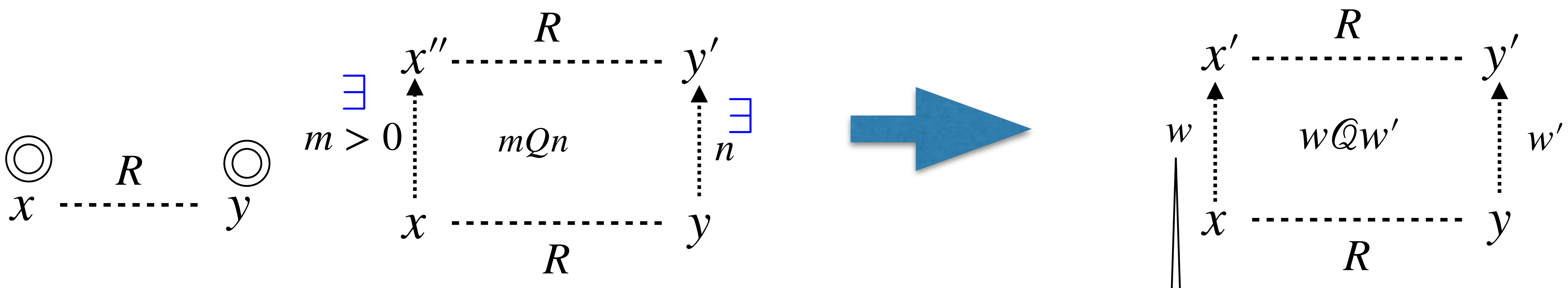
Addendum VI:
Preorder-Constrained Simulation for Deterministic Unlabeled Systems

[M., PhD thesis]

Definition (when no deadend):

Let $Q \subseteq \mathbb{N} \times \mathbb{N}$ be a preorder. A ***Q-constrained simulation*** from $(c : X \to X, F_1 \subseteq X)$ to $(d : Y \to Y, F_2 \subseteq Y)$ is a relation $R \subseteq X \times Y$ such that

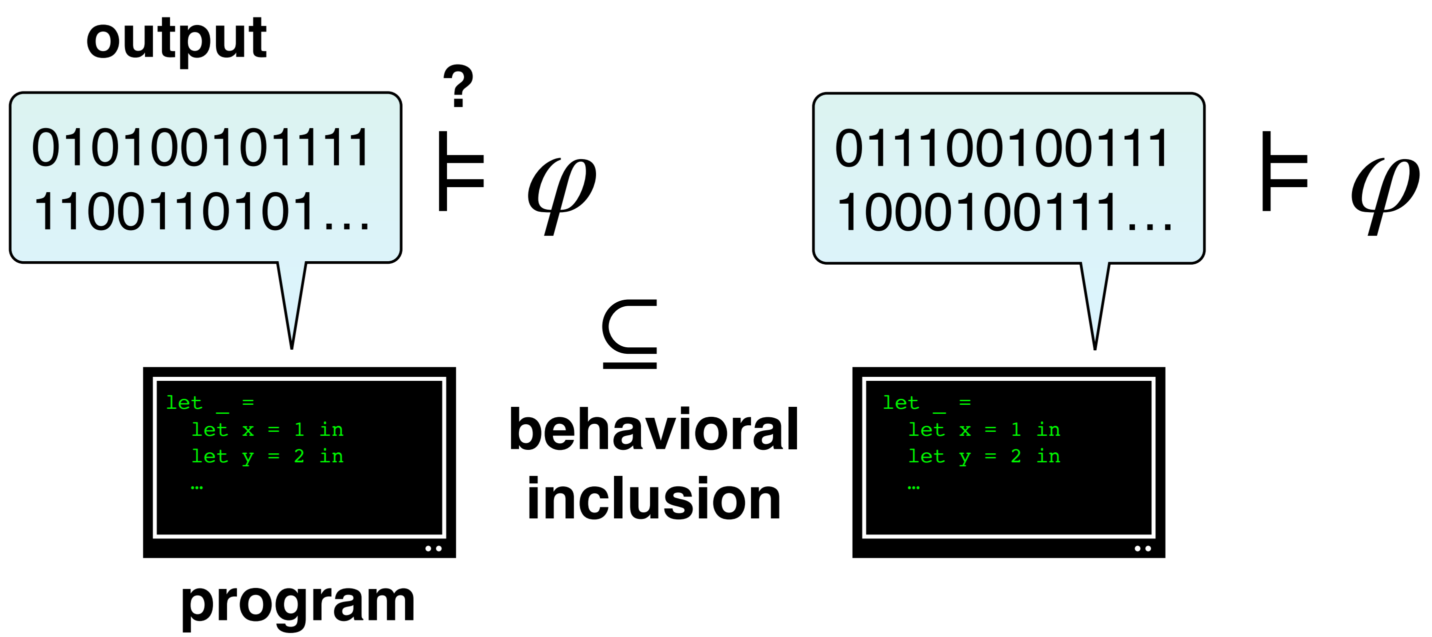
$$\forall (x, y) \in R \,.\quad \begin{array}{l} \text{- } x \in F_1 \implies y \in F_2 \\ \text{- } \exists m > 0, n \in \mathbb{N} \,.\, x \underbrace{\to \cdots \to}_{m} x'', y \underbrace{\to \cdots \to}_{n} y', x'' R y' \text{ and } mQn \end{array}$$



- Goal: generalization to nondeterministic automata
- Difficulty: both m and n are chosen by ∃

We wish:
nondeterminism is resolved by ∀,
length is determined by ∃

Addendum VII: Verification via Behavioral Inclusion



output

010100101111
1100110101…

$\overset{?}{\models} \varphi$

```
let _ =
  let x = 1 in
  let y = 2 in
  …
```

program

$\subseteq$

behavioral inclusion

011100100111
1000100111…

$\models \varphi$

```
let _ =
  let x = 1 in
  let y = 2 in
  …
```

- Behavioral inclusion between nondeterministic automata
    - Two formalizations: **trace inclusion** and **simulation**