A coalgebraic classification of power series

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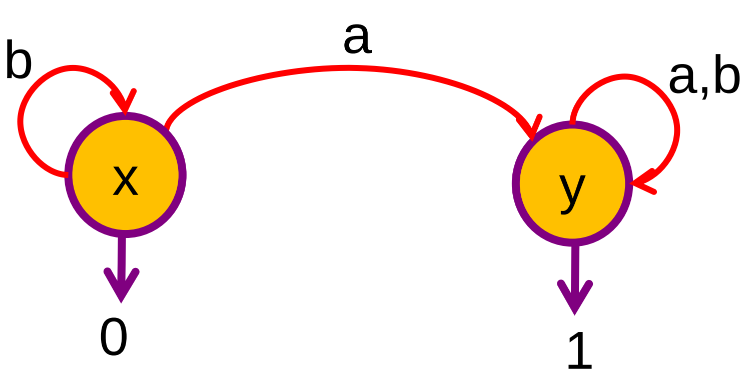
Background

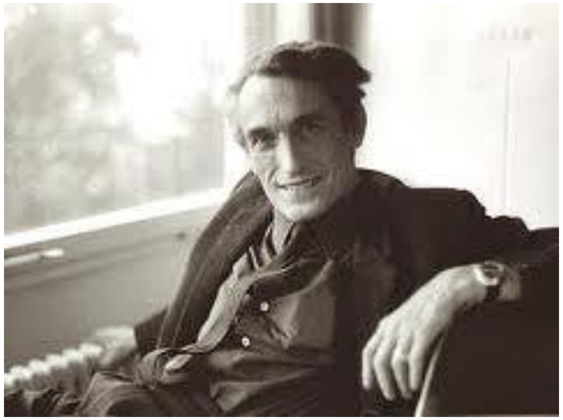
Kleene 1956:

Finite automata are regular expressions



b*a(a+b)*

Background



Schützenberger 1961:

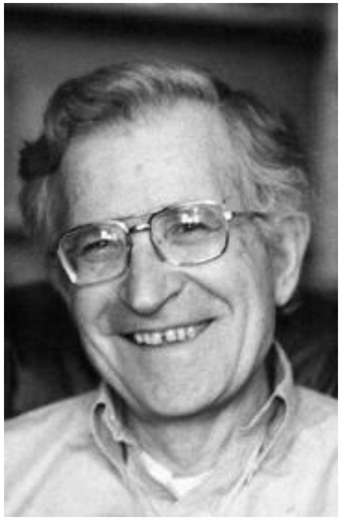
Languages are non commutative series and operations on regular expressions are rationals.

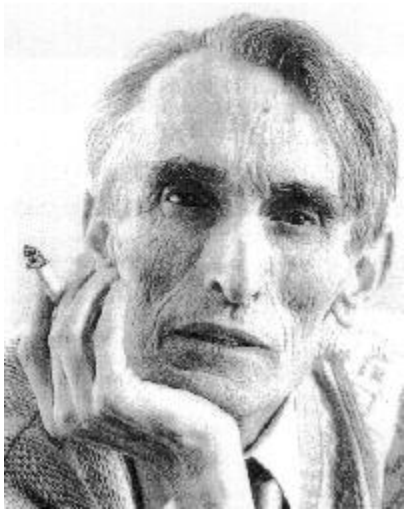
$L = \{\varepsilon, ab, aab\}$ $\qquad 1 + ab + aab$

Addition = union
Subtraction = take the coefficients from a field
Product = concatenation
Division = star

$$a^* = 1 + a + a^2 + a^3 + \ldots = \frac{1}{1 - a}$$

Background





Chomsky, Schützenberger 1963:

Algebraic system of equations have power series as solutions.

$S \to SS \mid aSb \mid bSa$

$$\begin{cases} x = xx + ay + bz \\ y = yb \\ z = za \end{cases}$$

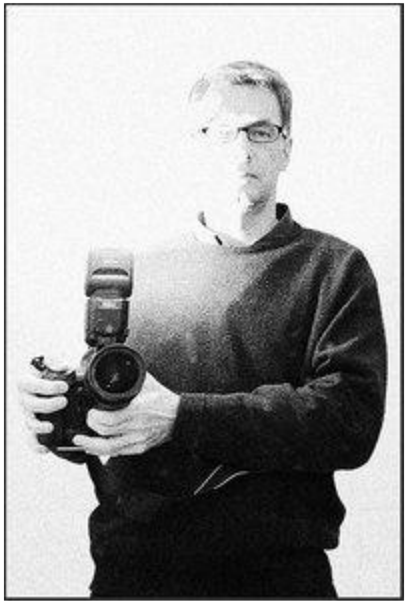
Background

Fliess1971:

Solutions of algebraic equations in one variable are algebraic streams.

$$1 - (X+2)\sigma + 4X\sigma^2 = 0$$

Background



Rutten1999:

Streams and power series are solutions of behavioral differential equations.

$\mathbf{o}(x) = 0$
$x' = x + 1$

$x = 0 \;\; 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \;\; 6 \;\; 7 \;\; \ldots$

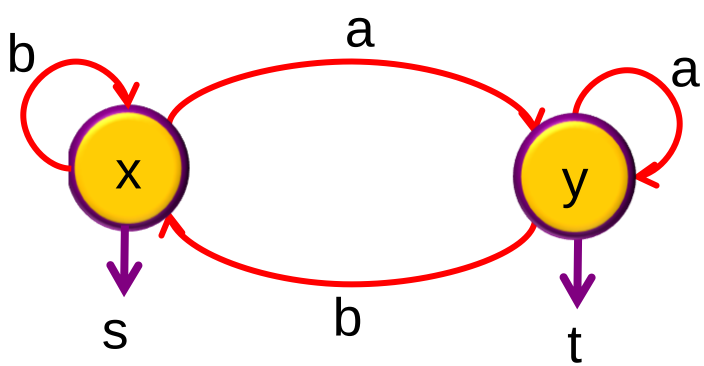
Starting point

Bonchi, Boreale, Milius, Rot, Rutten, Silva, Winter

Part I

Finite state power series

Moore automata



$$X \to S \times X^A$$

$$x \xrightarrow{a} y \xrightarrow{b} x \rightarrow s$$

$$x \xrightarrow{a} y \xrightarrow{a} y \xrightarrow{a} y \rightarrow t$$

$$S^{A^*}$$

Moore automata, coalgebraically

$$\begin{tikzcd}
X \arrow[r, dashed, "{[[-]]}"] \arrow[d, "f"'] & S^{A^*} \arrow[d, "\cong"] \\
S \times X^A \arrow[r, "{S \times [[-]]^A}"'] & S \times (S^{A^*})^A
\end{tikzcd}$$

Behavioral equation

$\mathbf{o}(x) = s$

$x_{\mathbf{a}} = y$

$\Longleftrightarrow$ $f(x) = \langle s, \phi \rangle$ and $\phi(a) = y$

(Formal) power series

Semirings $(\mathbb{S},+,0,\cdot,1)$

Power series $A^* \to \mathbb{S} = \mathbb{S}\langle\langle A^* \rangle\rangle$

$$\sigma = s_0 + s_1 a + s_2 b + s_3 a^2 + s_4 ab + s_5 ba + \ldots = \sum_{w \in A^*} (\sigma, w)\, w$$

Polynomials $A^* \to_{fs} \mathbb{S} = \mathbb{S}\langle A^* \rangle$

Examples

$(\mathbb{B},\vee,0,\wedge,1)$	Boolean semiring
$(\mathbb{F}_2,+,0,\cdot,1)$	Binary field (arithmetic modulo 2)
$(\mathbb{N},+,0,\cdot,1)$	Natural numbers
$(\mathbb{R},+,0,\cdot,1)$	Real numbers
$(\mathbb{R}^\infty,\min,\infty,+,0)$	Tropical semiring

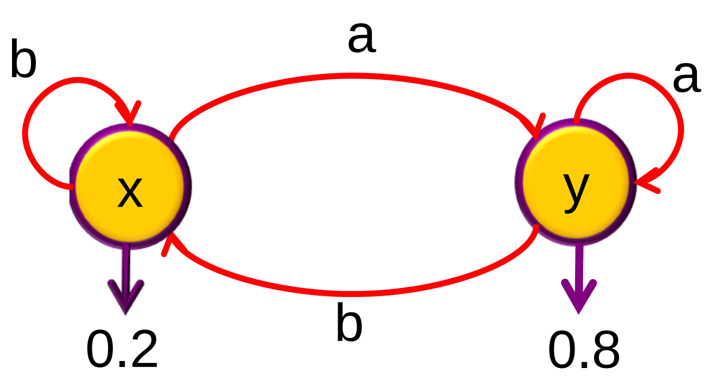
Finite state power series

Finite

$$
\begin{tikzcd}
X \arrow[r, dashed, "{[\![-]\!]}"] \arrow[d, "f"'] & \mathbb{S}^{fs}\langle\langle A^* \rangle\rangle \arrow[r, dashed, tail] \arrow[d, "\cong"] & \mathbb{S}\langle\langle A^* \rangle\rangle \arrow[d, "\cong"] \\
\mathbb{S} \times X^A \arrow[r] & \mathbb{S} \times \mathbb{S}^{fs}\langle\langle A^* \rangle\rangle^A \arrow[r] & \mathbb{S} \times (\mathbb{S}\langle\langle A^* \rangle\rangle)^A
\end{tikzcd}
$$

Behaviour of *finite* Moore automata

Example



Behavioral equations

$\mathbf{o}(x) = 0.2$	$\mathbf{o}(y) = 0.8$
$x_{\mathbf{a}} = y$	$y_{\mathbf{a}} = y$
$x_{\mathbf{b}} = x$	$y_{\mathbf{b}} = x$

$(\mathbb{R},+,0,\cdot,1)$ Real numbers

$[[x]](\varepsilon) = 0.2$ $[[x]](wb) = 0.8$ $[[x]](wa) = 0.8$

$[[x]] = 0.2 + 0.8a + 0.2\,b + 0.8a^2 + 0.2ab + 0.8ba + 0.2b^2 +$

Properties of $\mathbb{S}^{\text{fs}}\langle\langle A^* \rangle\rangle$

- Decidable bisimulation

  Implemented in CIRC [B.,Caltais, Goriac, Lucanu, Rutten,Silva 10]

- Several algorithms for minimization

  Partition-refinement [Bonchi et al.]

  Brzozowski minimization: [Bonchi,B., Rutten, Silva 09] + Hansen

- Complete axiomatization [Silva,B., Rutten 09]

Some closure properties of $\mathbb{S}^{fs}\langle\langle A^* \rangle\rangle$

$\mathbf{o}(x) = s_1$	$\mathbf{o}(y) = s_2$	…
$x_{\mathbf{a}} = x_1$	$y_{\mathbf{a}} = y_1$	…
$x_{\mathbf{b}} = \ldots$	$y_{\mathbf{b}} = \ldots$	…

Sum of x and y

$\mathbf{o}(\underline{x+y}) = s_1 + s_2$

$\underline{x+y}_{\mathbf{a}} = \underline{x_1 + y_1}$

$\underline{x+y}_{\mathbf{b}} = \ldots$

Product of x and y

$\mathbf{o}(\underline{x \otimes y}) = s_1 \cdot s_2$

$\underline{x \otimes y}_{\mathbf{a}} = \underline{x_1 \otimes y_1}$

$\underline{x \otimes y}_{\mathbf{b}} = \ldots$

Part II

Rational power series

Weighted automata

$TX \qquad t ::= x \mid s\cdot t \mid t + t$

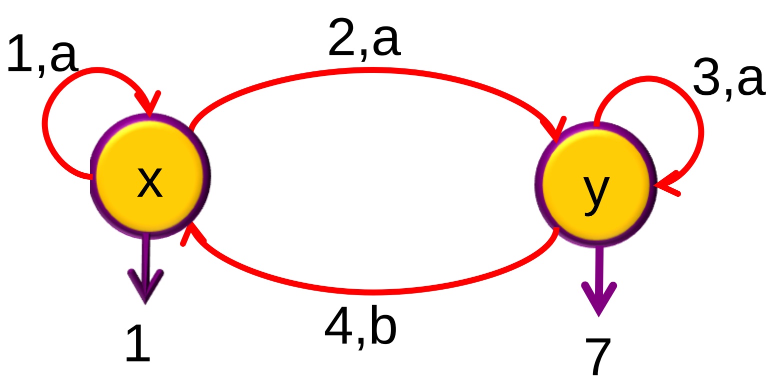
$$
\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & TX \arrow[r, dashed, "{[[-]]}"] \arrow[dl] & \mathbb{S}\langle\langle A^* \rangle\rangle \arrow[d, "\cong"] \\
\mathbb{S} \times TX^A \arrow[rr, "\mathbb{S} \times {[[-]]}^A"'] & & \mathbb{S} \times (\mathbb{S}\langle\langle A^* \rangle\rangle)^A
\end{tikzcd}
$$

Distributive law induced by $\Sigma F \to FT$

$s_1 \cdot (s_2, \phi) \longmapsto (s_1 \cdot s_2,\ \lambda a. s_1 \cdot \phi(a))$

$(s_1, \phi_1) + (s_2, \phi_2) \longmapsto (s_1 + s_2,\ \lambda a. \phi_1(a) + \phi_2(a))$

Weighted automata, classically



$L(x) \in \mathbb{N}\langle\langle A^* \rangle\rangle$

$x \xrightarrow{1,a} x \xrightarrow{2,a} y \xrightarrow{4,b} x \rightarrow 1$

$x \xrightarrow{2,a} x \xrightarrow{3,a} y \xrightarrow{4,b} x \rightarrow 1$

$L(x)(aab) = 8 + 24 = 32$

$x \xrightarrow{2,a} x \xrightarrow{4,b} y \rightarrow 1$

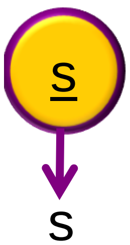
$L(x)(ab) = 8$

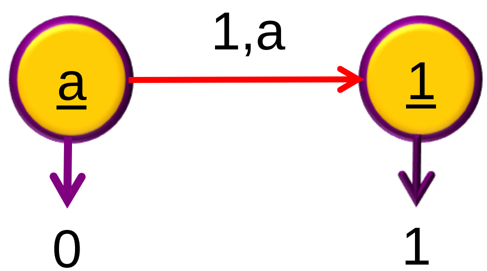
Two examples …

$\mathbf{o}(\underline{s}) = s$	$\mathbf{o}(\underline{a}) = 0$
$\underline{s}_{\mathbf{a}} = 0 \cdot \underline{s}$	$\underline{a}_{\mathbf{a}} = \underline{1}$
$\underline{s}_{\mathbf{b}} = 0 \cdot \underline{s}$	$\underline{a}_{\mathbf{b}} = \underline{0}$

$s \in \mathbb{S}$, $a \in A$

$$\begin{array}{c} X \\ \downarrow \\ \mathbb{S} \times TX^{A} \end{array}$$





$[[\underline{s}]] = s + 0a + 0b + \ldots$

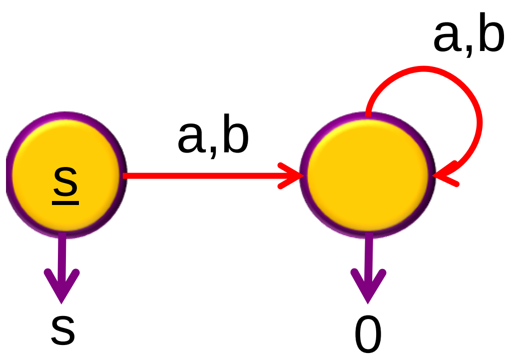
$[[\underline{a}]] = 0 + 1a + 0b + \ldots$

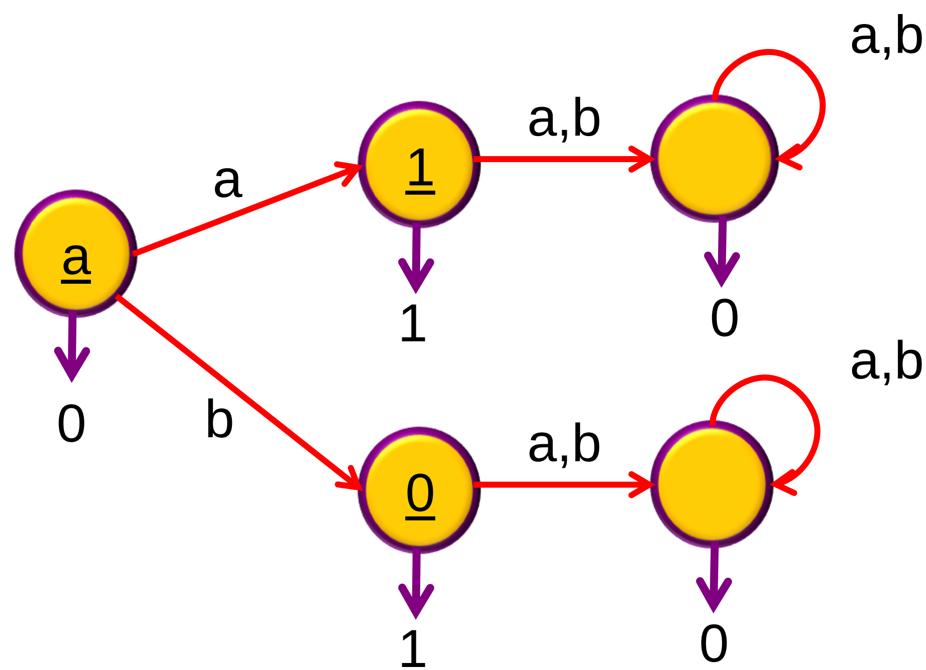
… and their Moore automata

TX

$\mathbb{S} \times TX^A$

$\mathbf{o}(\underline{s}) = s$	$\mathbf{o}(\underline{a}) = 0$
$\underline{s}_{\mathbf{a}} = 0 \cdot \underline{s}$	$\underline{a}_{\mathbf{a}} = \underline{1}$
$\underline{s}_{\mathbf{b}} = 0 \cdot \underline{s}$	$\underline{a}_{\mathbf{b}} = \underline{0}$

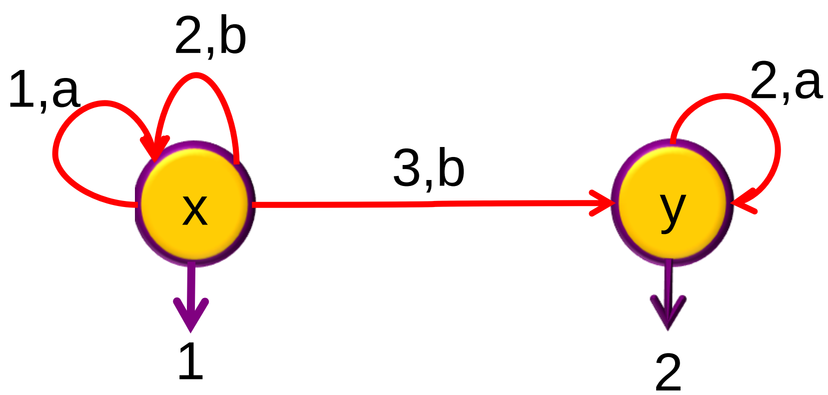




A larger example …

$\mathbf{o}(x) = 1$	$\mathbf{o}(y) = 2$
$x_{\mathbf{a}} = x$	$y_{\mathbf{a}} = 2 \cdot y$
$x_{\mathbf{b}} = 2 \cdot x + 3 \cdot y$	$y_{\mathbf{b}} = 0 \cdot y$

$$\begin{tikzcd} X \arrow[d] \\ \mathbb{N} \times TX^A \end{tikzcd}$$



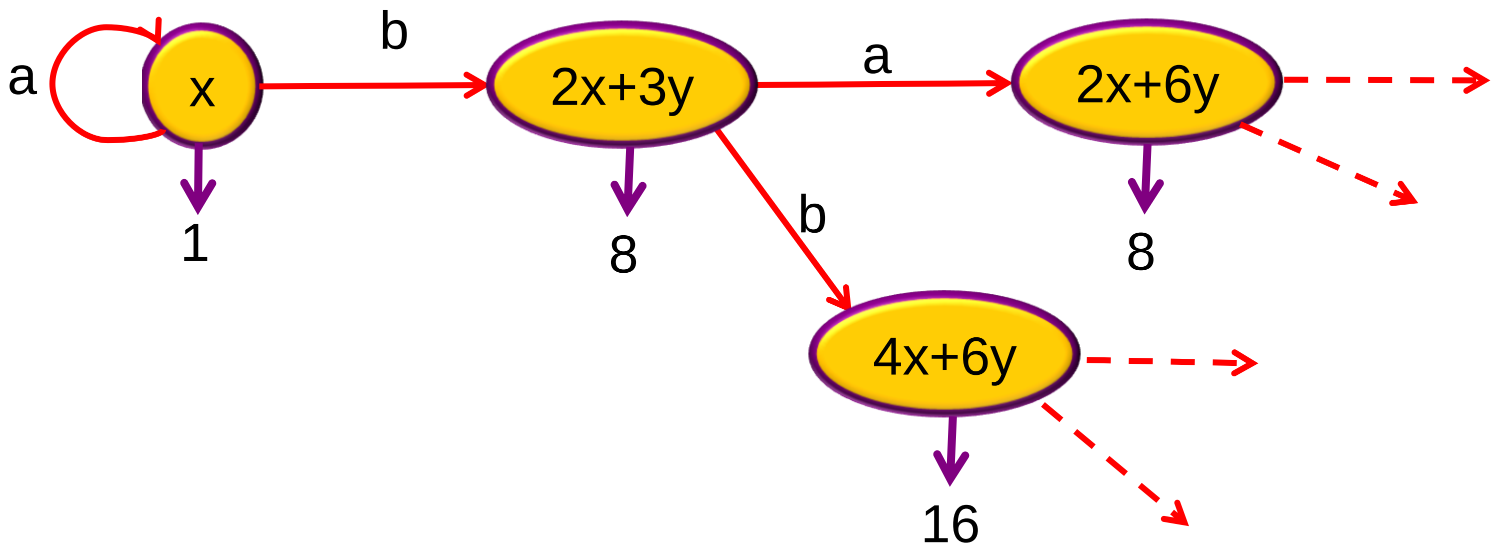
$$[[\underline{x}]] = 1 + 1a + 8b + 8ab + 14ba + \ldots$$

… and its Moore automaton

$\mathbf{o}(x) = 1$	$\mathbf{o}(y) = 2$
$x_{\mathbf{a}} = x$	$y_{\mathbf{a}} = 2 \cdot y$
$x_{\mathbf{b}} = 2 \cdot x + 3 \cdot y$	$y_{\mathbf{b}} = 0 \cdot y$

TX

$\mathbb{N} \times TX^A$



Few equalities

- $(TX, +, \underline{0})$ is a commutative monoid
- $s \cdot$ is a *scalar* product

$(s_1 + s_2)\cdot t \sim s_1\cdot t + s_2\cdot t$ $\qquad$ $s\cdot(t_1 + t_2) \sim s\cdot t_1 + s\cdot t_2$

$0\cdot t \sim \underline{0}$ $\qquad$ $s\cdot\underline{0} \sim \underline{0}$

$1\cdot t \sim t$ $\qquad$ $(s_1\cdot s_2)\cdot t \sim s_1\cdot(s_2\cdot t)$

Rational power series

Finite

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & TX \arrow[r, dashed, "{[[-]]}"] \arrow[dl] & \mathbb{S}^{rat}\langle\langle A^* \rangle\rangle \arrow[r, dashed, tail] \arrow[d, "\cong"] & \mathbb{S}\langle\langle A^* \rangle\rangle \arrow[d, "\cong"] \\
\mathbb{S} \times TX^A \arrow[rr] & & \mathbb{S} \times \mathbb{S}^{rat}\langle\langle A^* \rangle\rangle^A \arrow[r] & \mathbb{S} \times (\mathbb{S}\langle\langle A^* \rangle\rangle)^A
\end{tikzcd}
$$

TX $\quad t ::= x \mid s \cdot t \mid t + t$

Properties of $\mathbb{S}^{\mathrm{rat}}\langle\langle A^* \rangle\rangle$

[Sakarovitch 03]

Decidable equivalence for many semirings [Ésik,Maletti 10]

- Yes: Natural numbers, any subsemiring of a field
- No: tropical semiring, regular languages

Complete axiomatization for many semirings

- For regular languages [Krob 90][Kozen 94]
- For proper semirings [Ésik, Kuich 12]
- Adding equations to those of bisimulation [B.,Milius,Silva 12]

Several minimization algorithms

[Schützenberger 61], [Berstel,Reutenauer 88], [Sakarovitch 06], [Mohri 09],[Bonchi,B.,Boreale, Rutten,Silva 11]

Part III

Context free power series

Context free behavioral equations

TX $\quad t ::= x \mid s \mid t + t \mid t \cdot t$

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & TX \arrow[rr, dashed, "{[\![-]\!]}"] \arrow[dl] & & \mathbb{S}\langle\langle A^* \rangle\rangle \arrow[d, "\cong"] \\
\mathbb{S} \times TX^A \arrow[rrr, "{\mathbb{S} \times [\![-]\!]^A}"'] & & & \mathbb{S} \times (\mathbb{S}\langle\langle A^* \rangle\rangle)^A
\end{tikzcd}
$$

Distributive law induced by $\Sigma(F \times id) \to FT$

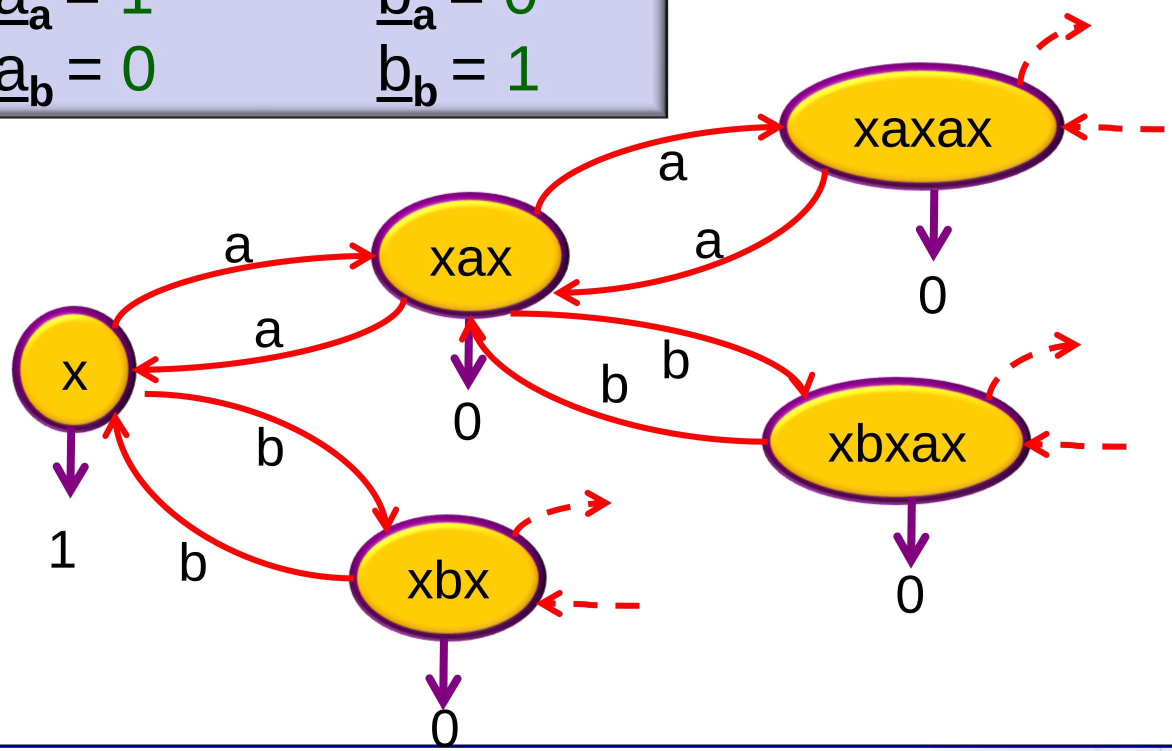
$s \longmapsto (s, \lambda a.0, s)$

$(s_1, \phi_1, \nu_1) \cdot (s_2, \phi_2, \nu_2) \longmapsto (s_1 \cdot s_2, \lambda a.\phi_1(a) \cdot \nu_2 + s_1 \cdot \phi_2(a))$

Example

Dyck language

$\mathbf{o}(x) = 1$	$\mathbf{o}(\underline{a}) = 0$	$\mathbf{o}(\underline{b}) = 0$
$x_{\mathbf{a}} = x \cdot \underline{a} \cdot x$	$\underline{a}_{\mathbf{a}} = 1$	$\underline{b}_{\mathbf{a}} = 0$
$x_{\mathbf{b}} = x \cdot \underline{b} \cdot x$	$\underline{a}_{\mathbf{b}} = 0$	$\underline{b}_{\mathbf{b}} = 1$



Context free power series

Finite

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & TX \arrow[r, dashed, "{[[-]]}"] \arrow[dl] & \mathbb{S}^{cf}\langle\langle A^* \rangle\rangle \arrow[r, dashed, tail] \arrow[d, "\cong"] & \mathbb{S}\langle\langle A^* \rangle\rangle \arrow[d, "\cong"] \\
\mathbb{S} \times TX^A \arrow[rr] & & \mathbb{S} \times \mathbb{S}^{cf}\langle\langle A^* \rangle\rangle^A \arrow[r] & \mathbb{S} \times (\mathbb{S}\langle\langle A^* \rangle\rangle)^A
\end{tikzcd}$$

TX $\quad t ::= x \mid s \mid t + t \mid t \cdot t$

$(TX, +, 0, \cdot, 1)$ is a semiring

- $(TX, +, 0)$ is a commutative monoid
- $(TX, \cdot, 1)$ is a monoid

Distributivity

$(t_1 + t_2)\cdot t_3 \sim t_1\cdot t_3 + t_2\cdot t_3 \qquad t_1\cdot(t_2 + t_3) \sim t_1\cdot t_2 + t_1\cdot t_3$

Annihilation

$0\cdot t \sim 0 \qquad\qquad t\cdot 0 \sim 0$

A simple closure property

TX $\quad t ::= x \mid s \mid t + t \mid t \cdot t$

$\mathbb{S}$ a field

$\mathbf{o}(x) = s$

$x_{\mathbf{a}} = t_1$

$x_{\mathbf{b}} = t_2$

…

$\mathbf{o}(y) = \ldots$

$y_{\mathbf{a}} = \ldots$

…

Lemma: $\quad x \cdot \frac{1}{x} = 1$

Inverse of x (if $s \neq 0$)

$\mathbf{o}(\frac{1}{x}) = s^{-1}$

$(\frac{1}{x})_{\mathbf{a}} = -s^{-1} \cdot t_1 \cdot \frac{1}{x}$

$(\frac{1}{x})_{\mathbf{b}} = -s^{-1} \cdot t_2 \cdot \frac{1}{x}$

…

Part IV

Algebraic power series

$\mathbb{S}$-algebraic systems

Finite

$x_i = p_i$ $\qquad p_i \in \mathbb{S}\langle(A+X)^*\rangle$ $\qquad x_i \in X$

$x = 2axb + by$
$y = by + b$

Proper = in each p_i occurs

1. no constant from $\mathbb{S}$
2. no single variable from X

Every proper $\mathbb{S}$-algebraic system has a solution in $\mathbb{S}\langle\langle A^*\rangle\rangle$

Not proper

$x = a + x$ and $x = 2 + xx$ $\qquad$ no solution in $\mathbb{N}\langle\langle A^*\rangle\rangle$

$x = x \cdot x$ $\qquad$ two solutions, 0 and 1

$\mathbb{S}$−algebraic power series $\mathbb{S}^{Alg}\langle\langle A^* \rangle\rangle$

$$\sigma \in \mathbb{S}\langle\langle A^* \rangle\rangle$$

$$\sigma = s_0 + \underbrace{s_1\, a + s_2\, b + s_3\, a^2 + s_4\, ab + s_5\, ba + \ldots}$$

Strong solution of a
proper $\mathbb{S}$-algebraic system

$\mathcal{S}^{\mathrm{Alg}}\langle\langle A^* \rangle\rangle = \mathcal{S}^{\mathrm{cf}}\langle\langle A^* \rangle\rangle$

$x = 2axb + by$
$y = by + b$

$\mathbf{o}(\underline{x}) = s_0$ | $\mathbf{o}(\underline{y}) = 0$ | $\mathbf{o}(\underline{b}) = 0$
$\underline{x}_{\mathbf{a}} = 2 \cdot x \cdot \underline{b}$ | $\underline{y}_{\mathbf{a}} = 0$ | $\underline{b}_{\mathbf{a}} = 0$
$\underline{x}_{\mathbf{b}} = y$ | $\underline{y}_{\mathbf{b}} = \underline{y} + 1$ | $\underline{b}_{\mathbf{b}} = 1$

Using Greibach normal form

$\underline{x} = a(\underline{y}+2) + 3b$
$\underline{y} = 4a(\underline{x}+1)$

$\mathbf{o}(x) = 1$ | $\mathbf{o}(y) = 2$
$x_{\mathbf{a}} = y$ | $y_{\mathbf{a}} = 4x$
$x_{\mathbf{b}} = 3$ | $y_{\mathbf{b}} = 0$

$\mathbb{S}^{cf}\langle\langle A^* \rangle\rangle$ is closed under reverse

$\mathbf{o}(x) = 1$ $\qquad$ $\mathbf{o}(\underline{b}) = 0$

$x_{\mathbf{a}} = x \cdot \underline{b}$ $\qquad$ $\underline{b}_{\mathbf{a}} = 0$

$x_{\mathbf{b}} = 0$ $\qquad$ $\underline{b}_{\mathbf{b}} = 1$

$\underline{x} = a(\underline{x}+1)\underline{b}$

$\underline{b} = b$

reverse

$x^R = \underline{b}^R(x^R+1)a$

$\underline{b}^R = b$

GNF

$x^R = bx^Ra + ba$

$\underline{a} = a$

$\mathbf{o}(\underline{x}^R) = 1$ $\qquad$ $\mathbf{o}(\underline{a}) = 0$

$\underline{x}^R{}_{\mathbf{a}} = 0$ $\qquad$ $\underline{a}_{\mathbf{a}} = 1$

$\underline{x}^R{}_{\mathbf{b}} = \underline{x}^R \cdot \underline{a} + \underline{a}$ $\qquad$ $\underline{a}_{\mathbf{b}} = 0$

Part V

Algebraic streams

Algebraic streams over $\mathbb{F}$

$A = \{ X \}$

$\mathbb{F}$ is a field

$\mathbb{F}\langle\langle A^* \rangle\rangle \cong \mathbb{N} \to \mathbb{F}$ i.e. streams

A stream $\sigma \in \mathbb{F}\langle A^* \rangle$ is algebraic if there are non-null polynomials $p_i \in \mathbb{F}\langle A^* \rangle$ such that

$$p_0 + p_1\sigma + p_2\,\sigma^2 + \ldots + p_n\sigma^n = 0$$

Algebraic streams are context free

Paper folding stream

σ = 1 1 0 1 1 0 0 1 1 1 0 0 …

$\mathbb{F}_2$

$1 - \sigma + X\sigma^2 = 0$

$\rightarrow$ $-\sigma' + \sigma^2 = 0$ Taking derivative

$\rightarrow$ $\sigma' = \sigma^2$

Behavioral equation

$\mathbf{o}(\sigma) = 1$
$\sigma' = \sigma \cdot \sigma$

Algebraic streams are context free

The Thue-Morse stream

$\sigma = 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ \ldots$

$\mathbb{F}_2$

$$X + (1+X^2)\sigma + (1+X+X^2+X^3)\sigma^2 = 0$$

Behavioral equations

$\mathbf{o}(x) = 0$	$\mathbf{o}(y) = 1$	$\mathbf{o}(z) = 1$	$\mathbf{o}(w) = 1$
$x' = y$	$y' = z$	$z' = z+y+x \cdot x \cdot w$	$w' = y \cdot w$

$\mathbb{F}$−algebraic $= \mathbb{F}^{cf}\langle\langle \{X\}^* \rangle\rangle$

In any field $\mathbb{F}$ and a singleton alphabet A, if a
well behaved stream is
algebraic
then
is context-free

[BRW 12]

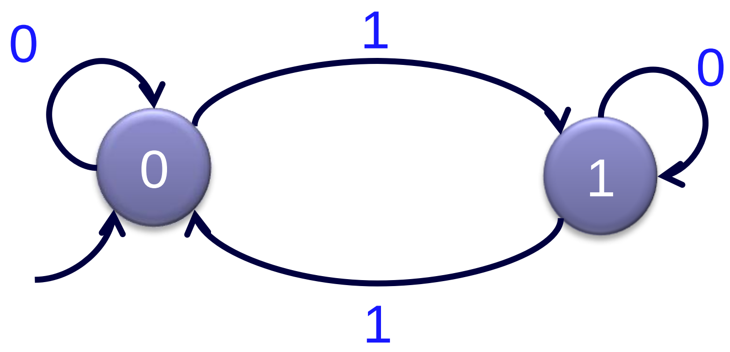
For a *perfect* field $\mathbb{F}$ and a singleton alphabet A,
a stream is
algebraic
if and only if
is $\mathbb{F}$−algebraic

[Fliess 71]

Part VI

Automatic streams

p-Automatic streams

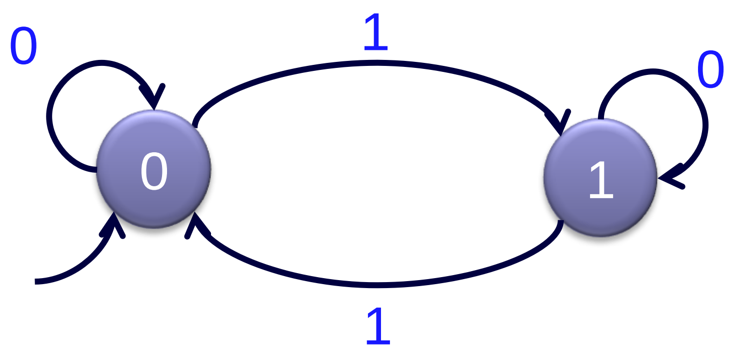


$\mathbb{F}_2$

The *n*-th term of the stream is the output of the state when the automata input is the digits of *n* in some fixed base *p*

[Allouche, Sallit 03]

Example: Thue-Morse again …



$\mathbb{F}_2$

Input	binary	output
0	0	0
1	1	1
2	10	1
3	11	0
4	100	1
5	101	0

Input	binary	output
6	101	0
7	111	1
8	1000	1
9	1001	0
10	1010	0
11	1011	1

p-automatic are context free

For any prime number p, a stream is
p-automatic
if and only if
is algebraic over $\mathbb{F}_p$

[Berstel,Retenauer 11]

The big picture

FS behavioral equations

Rat behavioral equations

Weighted CFG in GNF = CF behavioral equations

$\mathbb{S}$-algebraic

semirings

power series

Perfect field

fields

streams

Algebraic over a field

p prime

p-automatic streams

Why?

- Clarifying the definitions
- Importance of weighted systems
- Towards the understanding of treatable subsets of the final coalgebra
- Algorithmically interesting