Weak bisimulations for coalgebras over ordered functors

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CMCS 2012

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Two equivalent definitions of weak bisimulation for LTS

Definition

Let Σ be a set of labels and let $\tau \in \Sigma$ be a silent transition label. Let $\langle A, \Sigma, \to \rangle$ be a labelled transition system. A relation $R \subseteq A \times A$ is a *weak bisimulation* if it satisfies the following condition. If $(a, b) \in R$ then

$a \stackrel{\tau^*}{\to} \circ \stackrel{\sigma}{\to} \circ \stackrel{\tau^*}{\to} a'$ if and only if $b \stackrel{\tau^*}{\to} \circ \stackrel{\sigma}{\to} \circ \stackrel{\tau^*}{\to} b'$ and $(a', b') \in R$ for $\sigma \neq \tau$,

$a \stackrel{\tau^*}{\to} a'$ if and only if $b \stackrel{\tau^*}{\to} b'$ and $(a', b') \in R$ for $\sigma = \tau$.

Two equivalent definitions of weak bisimulation for LTS

Definition

Let Σ be a set of labels and let $\tau \in \Sigma$ be a silent transition label. Let $\langle A, \Sigma, \rightarrow \rangle$ be a labelled transition system. A relation $R \subseteq A \times A$ is a *weak bisimulation* if it satisfies the following condition. If $(a, b) \in R$ then

$$\text{for } \sigma \neq \tau \text{ if } a \xrightarrow{\sigma} a' \text{ then } b \xrightarrow{\tau^*} \circ \xrightarrow{\sigma} \circ \xrightarrow{\tau^*} b' \text{ and } (a', b') \in R,$$

$$\text{for } \sigma = \tau \text{ if } a \xrightarrow{\tau} a' \text{ then } b \xrightarrow{\tau^*} b' \text{ and } (a', b') \in R,$$

$$\text{for } \sigma \neq \tau \text{ if } b \xrightarrow{\sigma} b'' \text{ then } a \xrightarrow{\tau^*} \circ \xrightarrow{\sigma} \circ \xrightarrow{\tau^*} a'' \text{ and } (a'', b'') \in R,$$

$$\text{for } \sigma = \tau \text{ if } b \xrightarrow{\tau} b'' \text{ then } a \xrightarrow{\tau^*} a'' \text{ and } (a'', b'') \in R.$$

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Ordered functors

Let Pos be the category of all posets and monotonic mappings. Note that there is a forgetful functor $U : \mathsf{Pos} \to \mathsf{Set}$ assigning to each poset $(X, \leqslant)$ the underlying set X and to each monotonic map $f : (X, \leqslant) \to (Y, \leqslant)$ the map $f : X \to Y$.

Definition

An ordered functor is a functor $F : \mathsf{Set} \to \mathsf{Pos}$.

This notion is nothing new!

Ordered functors

Let Pos be the category of all posets and monotonic mappings. Note that there is a forgetful functor $U : \mathsf{Pos} \to \mathsf{Set}$ assigning to each poset $(X, \leqslant)$ the underlying set X and to each monotonic map $f : (X, \leqslant) \to (Y, \leqslant)$ the map $f : X \to Y$.

Definition

An ordered functor is a functor $F : \mathsf{Set} \to \mathsf{Pos}$.

This notion is nothing new! To any ordered functor F we assign the composition $\bar{F} = U \circ F$. We identify the ordered functor $F : \mathsf{Set} \to \mathsf{Pos}$ with $\bar{F} = U \circ F : \mathsf{Set} \to \mathsf{Set}$ and write F to denote both F and $\bar{F}$.

Ordered functors

Example

The powerset endofunctor $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ can be considered an ordered functor $\mathcal{P} : \mathsf{Set} \to \mathsf{Pos}$ which assigns to any set X the poset $(\mathcal{P}(X), \subseteq)$ and to any map $f : X \to Y$ the order preserving map $\mathcal{P}(f)$.

Ordered functors

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The powerset endofunctor $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ can be considered an ordered functor $\mathcal{P} : \mathsf{Set} \to \mathsf{Pos}$ which assigns to any set X the poset $(\mathcal{P}(X), \subseteq)$ and to any map $f : X \to Y$ the order preserving map $\mathcal{P}(f)$.

Example

Fix a nonempty Σ. Define $\mathcal{G}_\Sigma : \mathsf{Set} \to \mathsf{Set}$ by:

$$\mathcal{G}_\Sigma X := \{\mu : \Sigma \times \mathcal{P}(X) \to [0,1]\}$$

. For any set $\mathcal{G}_\Sigma X$ introduce the order $\leqslant_{\mathcal{G}_\Sigma X}$ given on $\mathcal{G}_\Sigma X$ as follows:

$$\mu_1 \leqslant_{\mathcal{G}_\Sigma X} \mu_2 \iff \mu_1(\sigma, X') \leqslant \mu_2(\sigma, X') \text{ for } (\sigma, X') \in \Sigma \times \mathcal{P}(X).$$

Ordered functors

Note that if we consider an ordered functor $F : \mathsf{Set} \to \mathsf{Pos}$ then we may introduce for any $X, Y \in \mathsf{Set}$ an order on the set $Hom(X, FY)$ as follows. For $f, g \in Hom(X, FY)$ put

$$f \leqslant g \overset{\text{def}}{\iff} f(x) \leqslant_{FY} g(x) \text{ for any } x \in X$$

Ordered functors

Note that if we consider an ordered functor $F : \mathsf{Set} \to \mathsf{Pos}$ then we may introduce for any $X, Y \in \mathsf{Set}$ an order on the set $Hom(X, FY)$ as follows. For $f, g \in Hom(X, FY)$ put

$$f \leqslant g \overset{\text{def}}{\iff} f(x) \leqslant_{FY} g(x) \text{ for any } x \in X$$

Given $f : X \to Y$, $\alpha : Y \to FZ$, $g : Z \to U$ and $\beta : Y \to FU$ an inequality $Fg \circ \alpha \leqslant \beta \circ f$ will be denoted by a diagram on the left and an equality $Fg \circ \alpha = \beta \circ f$ will be denoted by a diagram on the right:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\leqslant"] & Y \arrow[d, "\beta"] \\ FZ \arrow[r, "Fg"'] & FU \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "="] & Y \arrow[d, "\beta"] \\ FZ \arrow[r, "Fg"'] & FU \end{tikzcd}$$

Coalgebraic operators and coalgebraic saturators

Let C be full subcategory of the category of F-coalgebras and homomorphisms between them which is closed under taking inverse images of homomorphisms.

Definition

A *coalgebraic operator* $\mathfrak{s}$ with respect to a class C is a functor $\mathfrak{s} : \mathsf{C} \to \mathsf{Set}_F$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "\mathfrak{s}"] \arrow[dr, "U"'] & \mathsf{Set}_F \arrow[d, "U"] \\
& \mathsf{Set}
\end{tikzcd}$$

Coalgebraic operators and coalgebraic saturators

Let C be full subcategory of the category of F-coalgebras and homomorphisms between them which is closed under taking inverse images of homomorphisms.

Definition

A *coalgebraic saturator* $\mathfrak{s}$ with respect to a class C is a coalgebraic operator $\mathfrak{s} : \mathsf{C} \to \mathsf{Set}_F$ for which

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\leqslant"] & B \arrow[d, "\mathfrak{s}\beta"] \\ FA \arrow[r, "Ff"'] & FB \end{tikzcd} \iff \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathfrak{s}\alpha"'] \arrow[dr, phantom, "\leqslant"] & B \arrow[d, "\mathfrak{s}\beta"] \\ FA \arrow[r, "Ff"'] & FB \end{tikzcd}$$

Lemma

Let $\mathfrak{s} : \mathsf{C} \to \mathsf{Set}_F$ be an operator w.r.t. a full subcategory C of Set_F and additionally let $\mathfrak{s}(\mathsf{C}) \subseteq \mathsf{C}$. Then $\mathfrak{s}$ is a saturator if and only if it satisfies the following three properties:

- $\alpha \leqslant \mathfrak{s}\alpha$ *for any coalgebra* $\langle A, \alpha \rangle \in \mathsf{C}$ *(extensivity),*
- $\mathfrak{s} \circ \mathfrak{s} = \mathfrak{s}$ *(idempotency),*
- *if* $Ff \circ \alpha \leqslant \beta \circ f$ *then* $Ff \circ \mathfrak{s}\alpha \leqslant \mathfrak{s}\beta \circ f$ *for any* $f : X \to Y$ *(monotonicity):*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\leqslant"] & B \arrow[d, "\beta"] \\ FA \arrow[r, "Ff"'] & FB \end{tikzcd} \implies \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathfrak{s}\alpha"'] \arrow[dr, phantom, "\leqslant"] & B \arrow[d, "\mathfrak{s}\beta"] \\ FA \arrow[r, "Ff"'] & FB \end{tikzcd}$$

Coalgebraic saturators: examples

Example

Let $\tau \in \Sigma$ be a silent transition label. For a coalgebra structure $\alpha : A \to \mathcal{P}(\Sigma \times A)$ we define its saturation $\mathfrak{s}\alpha : A \to \mathcal{P}(\Sigma \times A)$ as follows. For an element $a \in A$ put

$$\mathfrak{s}\alpha(a) :=$$

$$\alpha(a) \cup \{(\tau, a') \mid a \stackrel{\tau^*}{\to} a'\} \cup \{(\sigma, a') \mid a \stackrel{\tau^*}{\to} \circ \stackrel{\sigma}{\to} \circ \stackrel{\tau^*}{\to} a' \text{ for } \sigma \neq \tau\}$$

$\mathfrak{s}$ is a coalgebraic saturator with respect to the class of all $\mathcal{P}(\Sigma \times (-))$-coalgebras.

Two approaches to defining weak bisimulation

From now on we will assume that $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ are members of C.

Definition

A relation $R \subseteq A \times B$ is said to be a *saturated weak bisimulation* between $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ provided that there is a structure $\gamma : R \to FR$ for which the following diagram commutes:

$$\begin{tikzcd} A \arrow[d, "\mathfrak{s}\alpha"'] \arrow[dr, phantom, "="] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma"] \arrow[dr, phantom, "="] & B \arrow[d, "\mathfrak{s}\beta"] \\ FA & FR \arrow[l, "F\pi_1"] \arrow[r, "F\pi_2"'] & FB \end{tikzcd}$$

Two approaches to defining weak bisimulation

Definition

A relation $R \subseteq A \times B$ is called a *weak bisimilation* provided that there is a structure $\gamma_1 : R \to FR$ and a structure $\gamma_2 : R \to FR$ for which:

- $\alpha \circ \pi_1 = F\pi_1 \circ \gamma_1$ and $F\pi_2 \circ \gamma_1 \leqslant \mathfrak{s}\beta \circ \pi_2$,
- $\beta \circ \pi_2 = F\pi_2 \circ \gamma_2$ and $F\pi_1 \circ \gamma_2 \leqslant \mathfrak{s}\alpha \circ \pi_1$.

$$
\begin{tikzcd}
A \arrow[d, "\alpha"'] \arrow[dr, phantom, "="] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma_1"] \arrow[dr, phantom, "\leqslant"] & B \arrow[d, "\mathfrak{s}\beta"] \\
FA & FR \arrow[l, "F\pi_1"] \arrow[r, "F\pi_2"'] & FB
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, "\mathfrak{s}\alpha"'] \arrow[dr, phantom, "\geqslant"] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma_2"] \arrow[dr, phantom, "="] & B \arrow[d, "\beta"] \\
FA & FR \arrow[l, "F\pi_1"] \arrow[r, "F\pi_2"'] & FB
\end{tikzcd}
$$

Examples

Example

For LTS's definition of a saturated weak bisimulation conicides with the first definition presented at the beginning. The definition of weak bisimulation from previous slide is exactly the 2nd definition of weak bisimulation presented at the beginning of this presentation.

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Weak bisimulation

Theorem

Let $R \subseteq A \times B$ be a standard bisimulation between $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$. Then R is a weak bisimilation between $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$.

Theorem

If a relation $R \subseteq A \times B$ is a weak bisimulation between $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ then $R^{-1} = \{(b, a) \mid (a, b) \in R\}$ is a weak bisimulation between $\langle B, \beta \rangle$ and $\langle A, \alpha \rangle$.

Weak bisimulation

Theorem

If all members of a family $\{R_i\}_{i \in I}$ *of relations* $R_i \subseteq A \times B$ *are weak bisimulations between* $\langle A, \alpha \rangle$ *and* $\langle B, \beta \rangle$ *then* $\bigcup_{i \in I} R_i$ *is also a weak bisimulation between* $\langle A, \alpha \rangle$ *and* $\langle B, \beta \rangle$*.*

Theorem

Let $F : \mathsf{Set} \to \mathsf{Set}$ *weakly preserve pullbacks and let* $\langle A, \alpha \rangle$, $\langle B, \beta \rangle$ *and* $\langle C, \delta \rangle$ *be* F*-coalgebras from the class* C*. Let* R_1 *be a weak bisimulation between* $\langle A, \alpha \rangle$ *and* $\langle B, \beta \rangle$ *and* R_2 *be a weak bisimulation between* $\langle B, \beta \rangle$ *and* $\langle C, \delta \rangle$*. Then*

$$R_1 \circ R_2 = \{(a, c) \mid \exists b \in B \text{ s.t. } (a, b) \in R_1 \text{ and } (b, c) \in R_2\}$$

is a weak bisimulation between $\langle A, \alpha \rangle$ *and* $\langle C, \delta \rangle$*.*

Weak bisimulation

Corollary

If $F : \mathsf{Set} \to \mathsf{Set}$ weakly preserves pullbacks then the greatest weak bisimulation on a coalgebra $\langle A, \alpha \rangle$ is an equivalence relation.

Saturated weak bisimulation

Theorem

Let $R \subseteq A \times B$ be a standard bisimulation between $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$. Then R is also a saturated weak bisimilation between $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$.

Theorem

A saturated weak bisimulation between $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ is defined as a standard bisimulation between saturated models $\langle A, \mathfrak{s}\alpha \rangle$ and $\langle B, \mathfrak{s}\beta \rangle$. Hence, any property true for standard bisimulation is also true for a saturated weak bisimulation.

Weak and saturated weak bisimulation

Theorem

Let $F : \mathsf{Set} \to \mathsf{Set}$ weakly preserve kernel pairs and let $R \subseteq A \times A$ be an equivalence relation which is a weak bisimulation on $\langle A, \alpha \rangle$. Then R is a saturated weak bisimulation on $\langle A, \alpha \rangle$.

We say that two elements $a, b \in A$ are weakly bisimilar, and write $a \approx_w b$ if there is a weak bisimulation $R \subseteq A \times A$ on $\langle A, \alpha \rangle$ for which $(a, b) \in R$. We say that a and b are saturated weakly bisimilar, and write $a \approx_{sw} b$, if there is a saturated weak bisimulation R on $\langle A, \alpha \rangle$ containing (a, b).

Corollary

Let $F : \mathsf{Set} \to \mathsf{Set}$ be a functor weakly preserving pullbacks. Then the relations $\approx_w$ and $\approx_{sw}$ are equivalence relations and

$$\approx_w \subseteq \approx_{sw} .$$

Saturated weak and weak bisimulation

Definition

We say that an ordered functor $F : \mathsf{Set} \to \mathsf{Pos}$ *preserves downsets* provided that for any $f : X \to Y$ and any $\vec{x} \in FX$ the following equality holds:

$$Ff(\vec{x}\downarrow) = Ff(\{\vec{x}' \in FX \mid \vec{x}' \leqslant \vec{x}\}) = Ff(\vec{x})\downarrow = \{\vec{y} \in FY \mid \vec{y} \leqslant Ff(\vec{x})\}.$$

Example

The powerset functor $\mathcal{P}$ preserves downsets.

Saturated weak and weak bisimulation

Theorem

Let $F : \mathsf{Set} \to \mathsf{Set}$ weakly preserve kernel pairs and preserve downsets. Let $R \subseteq A \times A$ be an equivalence relation which is a saturated weak bisimulation on $\langle A, \alpha \rangle$. Then R is a weak bisimulation on $\langle A, \alpha \rangle$.

Saturated weak bisimulation which is not weak bisimulation

Define a functor $F : \mathsf{Set} \to \mathsf{Set}$ on sets by

$$FX = (\{a\} \times X + \{b\} \times X^2)/\Theta + \{\perp\}$$

where Θ is the smallest equivalence relation on $\{a\} \times X + \{b\} \times X^2$ satisfying for any $x \in X$:

$$(b, x, x) \;\Theta\; (a, x).$$

We define F on morphisms in a natural way. For any set X introduce a partial order $\leqslant$ on FX as the smallest partial order satisfying

$$\perp \leqslant (a, x)/\Theta \text{ for any } x \in X.$$

The order $\leqslant$ is well defined and makes the functor F an ordered functor. The functor F does not preserve downsets.

Saturated weak bisimulation which is not weak bisimulation

For any F-coalgebra $\langle A, \alpha \rangle$ define an operator $\mathfrak{s}\alpha : A \to FA$ by

$$\mathfrak{s}\alpha(x) = \begin{cases} \alpha(x) & \text{if } \alpha(x) \neq \perp, \\ (a, x)/\Theta & \text{otherwise.} \end{cases}$$

The operator $\mathfrak{s} : \mathrm{Set}_F \to \mathrm{Set}_F$ is a coalgebraic saturator with respect to the class of all F-coalgebras.

Saturated weak bisimulation which is not weak bisimulation

For any F-coalgebra $\langle A, \alpha \rangle$ define an operator $\mathfrak{s}\alpha : A \to FA$ by

$$\mathfrak{s}\alpha(x) = \begin{cases} \alpha(x) & \text{if } \alpha(x) \neq \perp, \\ (a, x)/\Theta & \text{otherwise.} \end{cases}$$

The operator $\mathfrak{s} : \mathrm{Set}_F \to \mathrm{Set}_F$ is a coalgebraic saturator with respect to the class of all F-coalgebras.
Now consider a set $A = \{x, y\}$ and define a structure $\alpha : A \to FA$ by $\alpha(x) = \perp$ and $\alpha(y) = (b, x, y)/\Theta$. We see that $x \approx_{sw} y$, but x and y are not weakly bisimilar.

Weak and saturated weak bisimulation

Corollary

Let $F : \mathsf{Set} \to \mathsf{Set}$ *weakly preserve pullbacks and preserve downsets. Then for any* F*-coalgebra* $\langle A, \alpha \rangle \in \mathsf{C}$ *the relations* $\approx_w$ *and* $\approx_{sw}$ *are equivalence relations and*

$$\approx_w = \approx_{sw} .$$

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Fully probabilistic processes

A fully probabilistic process [Baier, Hermanns] is a tuple (A, Σ, P), where A is a set of *states*, Σ is an non-empty set called *alphabet* and $P : A \times \Sigma \times A \to [0, 1]$ is a function such that for any $a \in A$ the sum $\sum_{(\sigma, a') \in \Sigma \times A} P(a, \sigma, a') = 1$.

Fully probabilistic processes

A fully probabilistic process [Baier, Hermanns] is a tuple (A, Σ, P), where A is a set of *states*, Σ is an non-empty set called *alphabet* and $P : A \times \Sigma \times A \to [0,1]$ is a function such that for any $a \in A$ the sum $\sum_{(\sigma,a') \in \Sigma \times A} P(a,\sigma,a') = 1$. Any fully probabilistic process (A, Σ, P) can be view as $\mathcal{G}_\Sigma$-coalgebra

$$\alpha(a) : \Sigma \times \mathcal{P}(A) \to [0,1]; (\sigma, A') \mapsto P(a, \sigma, A').$$

Let *FPP* be the class of all fully probabilistic processes and homomorphisms between them in the category of all $\mathcal{G}_\Sigma$-coalgebras.

Remark

There are some problems with FPPs! Namely, the very natural operator is not a saturator...

We can put them only into one part of the setting.

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Terminal object in Set_F^w

Let Set_F^w denote the category of all F-coalgebras as objects and all maps $f : A \to B$ between $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ for which the relation $\{(a, f(a) \mid a \in A\}$ is a weak bisimulation between $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ as morphisms.

Theorem

Let F weakly preserve pullbacks. If $\langle T, \tau\rangle$ is the terminal object in Set_F then the greatest subcoalgebra $\langle T_s, \tau_s\rangle$ of $\langle T, \tau\rangle$ closed under saturation, i.e. $\mathfrak{s}\tau_s = \tau_s$ is the terminal object in Set_F^w.

Weak coinduction principle

Weak coinduction principle

Let F weakly preserve pullbacks. Let $\langle A, \alpha \rangle$ be any F-coalgebra and let $[\![-]\!]^w_\alpha$ denote the unique weak homomorphism from $\langle A, \alpha \rangle$ to $\langle T_s, \tau_s \rangle$. For two elements $a, b \in A$ we have

$$a \approx_w b \iff [\![a]\!]^w_\alpha = [\![b]\!]^w_\alpha$$

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How to define a saturation?

Let $F : \mathrm{Set} \to \mathrm{csLAT}$. Let T be a subfunctor of F which is a partially ordered monad with the unit $\eta : \mathcal{I}d \implies T$ and product $\mu : T^2 \implies T$. Consider two natural transformations

$$\lambda_1 : TF \implies F \quad \text{and} \quad \lambda_2 : FT \implies F.$$

Example

For the LTS functor $\mathcal{P}(\Sigma \times \mathcal{I}d)$ and the invisible transition $\tau \in \Sigma$ consider $T = \mathcal{P}(\{\tau\} \times \mathcal{I}d)$. It is a partially ordered monad since $T \approx \mathcal{P}$ and

$$\lambda_{1X} : TFX \to FX; (\tau, S') \mapsto S',$$

$$\lambda_{2X} : FTX \to FX; (\sigma, \{\tau\} \times S')) \mapsto (\sigma, S')$$

How to define a saturation?

Finally consider a *reductor* $\mathfrak{r} : \mathsf{Set}_F \to \mathsf{Set}_F$ which is an coalgebraic operator satisfying:

- for any F-coalgebra $\langle A, \alpha\rangle$ the coalgebra $\langle A, \mathfrak{r}\alpha\rangle$ is a T-coalgebra,
- $\mathfrak{r}\alpha \leqslant \alpha$
- $\mathfrak{r} \circ \mathfrak{r} = \mathfrak{r}$
- if $Ff \circ \alpha \leqslant \beta \circ f$ then $Ff \circ \mathfrak{r}\alpha \leqslant \mathfrak{r}\beta \circ f$ for any $f : X \to Y$ (monotonicity):

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\leqslant"] & B \arrow[d, "\beta"] \\ FA \arrow[r, "Ff"'] & FB \end{tikzcd} \implies \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathfrak{r}\alpha"'] \arrow[dr, phantom, "\leqslant"] & B \arrow[d, "\mathfrak{r}\beta"] \\ FA \arrow[r, "Ff"'] & FB \end{tikzcd}$$

Example

For LTS coalgebra $\langle A, \alpha\rangle$ and $a \in A$ consider $\mathfrak{r}\alpha(a) := \{(\tau, a') \mid (\tau, a') \in \alpha(a)\}$.

The saturation algebra

For any $\alpha, \alpha' : A \to FA$ define the following operations:

$$\begin{aligned}
&\alpha + \alpha' = \alpha \vee \alpha', \\
&\alpha \triangleright \alpha' := \lambda_{1A} \circ T\alpha \circ \mathfrak{r}\alpha', \\
&\alpha \triangleleft \alpha' := \lambda_{2A} \circ F\mathfrak{r}\alpha \circ \alpha', \\
&0 := \perp, \\
&1 := \eta_A, \\
&\alpha^* := \min\{\beta \mid \beta = 1 + \beta \triangleleft \alpha + \alpha \triangleright \beta\}.
\end{aligned}$$

We get a saturation algebra

$$(Hom(A, FA), +, \triangleleft, \triangleright, \mathfrak{r}, (-)^*, 0, 1).$$

Thank you for your attention!

Support

This work has been supported by the European Union in the framework of European Social Fund through the Warsaw University of Technology Development Programme.