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On Finitary Functors and Their Presentation

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Why finitary functors are interesting

Finitary functors F (on sets) are important in algebra and coalgebra

initial algebras $\mu F = \operatorname{colim}_{n < \omega} (0 \rightarrow F0 \rightarrow \dots \rightarrow Fn \rightarrow \dots)$
(J. Adámek 1974)

final coalgebras $\nu F = \lim_{n < \omega + \omega} (1 \leftarrow F1 \leftarrow \dots \leftarrow F^\omega 1 \leftarrow F^{\omega+1} 1 \leftarrow \dots)$
(J. Worrell 1999)

have presentation by $(\Sigma, E) : \quad F = H_\Sigma/E$

(J. Adámek & V. Trnková 1990) F -algebras = Σ -algebras satisfying E

BUT Set is not enough for many applications.

Our results.

1. Generalize to $F : \mathcal{A} \rightarrow \mathcal{A}$, where \mathcal{A} locally finitely presentable category.

Application of G.M. Kelly & A.J. Power 1993

Related to: Bonsangue & Kurz (2006); Kurz & Rosicky (2006); Kurz & Velebil (2011)

2. Hausdorff functor $\mathcal{H} : \text{CMS} \rightarrow \text{CMS} : \quad \text{finitary} + \text{presentation}$

Strengthening of: van Breugel, Hermida, Makkai, Worrell (2007)

Locally finitely presentable (lfp) categories

„Definition.“ \mathcal{A} locally finitely presentable : \iff

\mathcal{A} has good notion of “finite object”



= finitely presentable objects

$F : \mathcal{A} \rightarrow \mathcal{B}$ **finitary** $\iff F$ determined by its action on $\mathcal{F} \hookrightarrow \mathcal{A}$



= preserving filtered colimits



\mathcal{F} = full subcategory of finitely presentable objects

Examples. Set, posets, graphs, groups, vector spaces
finitary varieties of algebras

\mathcal{A} lfp, \mathcal{C} small $\implies \mathcal{A}^{\mathcal{C}}, \text{Fin}(\mathcal{A}, \mathcal{A}), \text{FinMnd}(\mathcal{A})$ lfp

Example: presentation of the finite power-set functor

$$\mathcal{P}_f : \text{Set} \rightarrow \text{Set} \quad \mathcal{P}_f X = \{ Y \mid Y \subseteq X, Y \text{ finite} \}$$

signature $\Sigma :$ $\Sigma_n = \{ \sigma_n \}$  $H_\Sigma X = \Sigma_0 + \Sigma_1 \times X + \Sigma_2 \times X^2 + \dots$
 $= X^*$

flat equations: $\sigma_\ell(x_0, \dots, x_{\ell-1}) = \sigma_k(y_0, \dots, y_{k-1})$

iff $\{x_0, \dots, x_\ell\} = \{y_0, \dots, y_k\}$

$$1 \xrightarrow[u]{u'} H_\Sigma \{0, \dots, k-1\}$$


$$\varepsilon : H_\Sigma \rightarrow \mathcal{P}_f \quad \varepsilon_X : (\sigma_i, x_0, \dots, x_{i-1}) \mapsto \{x_0, \dots, x_{i-1}\}$$

is universal

From Set to Ifp categories

Following Kelly & Power (1993)

signature $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$

$\Sigma : |\mathcal{F}| \rightarrow \mathcal{A}$

$$H_\Sigma X = \Sigma_0 + \Sigma_1 \times X + \Sigma_2 \times X^2 + \dots$$

$$H_\Sigma X = \coprod_{k \in \mathcal{F}} \mathcal{A}(k, X) \bullet \Sigma_k$$

flat equation $1 \xrightleftharpoons[u]{u'} H_\Sigma k$

$$n \xrightleftharpoons[u]{u'} H_\Sigma k \quad n, k \in \mathcal{F}$$

Construction of $F : \mathcal{A} \rightarrow \mathcal{A}$ presented by $\Sigma, \{n_i \xrightleftharpoons[u_i]{u'_i} H_\Sigma k_i \mid i \in I\}$

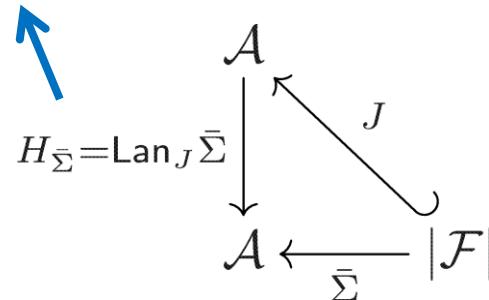
1. form the signature $\bar{\Sigma}_k = \coprod_{i \in I, k_i=k} n_i \quad \forall k \in \mathcal{F}$

2. copairing of all equations

$$\frac{\bar{\Sigma}_k \xrightarrow{[u_i]} H_\Sigma k \quad \forall k}{H_{\bar{\Sigma}} \xrightarrow{\bar{u}} H_\Sigma}$$

3. coequalizer

$$H_{\bar{\Sigma}} \xrightarrow{\bar{u}} H_\Sigma \xrightarrow{\varepsilon} F$$



Example in posets

signature $\Sigma_n = \begin{cases} 1 & \text{if } n = \downarrow \\ 0 & \text{else} \end{cases}$



one binary operation $\sigma(x, y)$
defined for $x \leq y$

flat equation $\sigma(x, x) = \sigma(y, y)$

specifies $G(X, \leq) = \{ (x, y) \in X^2 \mid x < y \} \cup \{ * \}$

for $f : (X, \leq) \rightarrow (Y, \sqsubseteq)$

$$Gf(x, y) = \begin{cases} (fx, fy) & \text{if } fx \sqsubseteq fy \\ \text{else} & \end{cases}$$

Finitary functors and presentations

Theorem. Let \mathcal{A} be an lfp category. \leftarrow locally λ -presentable category

$$F : \mathcal{A} \rightarrow \mathcal{A} \text{ finitary} \iff F \text{ given by a finitary presentation}$$

\nearrow
 λ -accessible

\nwarrow
 λ -ary

Proof. $\Leftarrow H_{\bar{\Sigma}} \xrightarrow[\bar{u}']{\bar{u}} H_{\Sigma} \xrightarrow{\varepsilon} F$ coequalizer of finitary functors

\Rightarrow canonical presentation of a finitary $F : \mathcal{A} \rightarrow \mathcal{A}$

$$\begin{array}{c} \text{signature } \Sigma_k = Fk \\ \hline \hline \Sigma_k \xrightarrow{\text{id}} Fk \quad \forall k \\ \hline \hline H_{\Sigma} = \text{Lan}_J \Sigma \xrightarrow{\varepsilon} F \end{array}$$

flat equations: all $n \xrightarrow[u]{u'} H_{\Sigma}k$ with $\varepsilon \cdot u = \varepsilon \cdot u'$

□

Theorem. Let $F : \mathcal{A} \rightarrow \mathcal{A}$ be presentable by (Σ, E) .

F -algebras = Σ -algebras satisfying flat equations in E

The Hausdorff functor

Non-determinism for systems with complete metric state space.

category CMS: complete metric spaces (X, d)

non-expanding maps $f : (X, d_X) \rightarrow (Y, d_Y)$

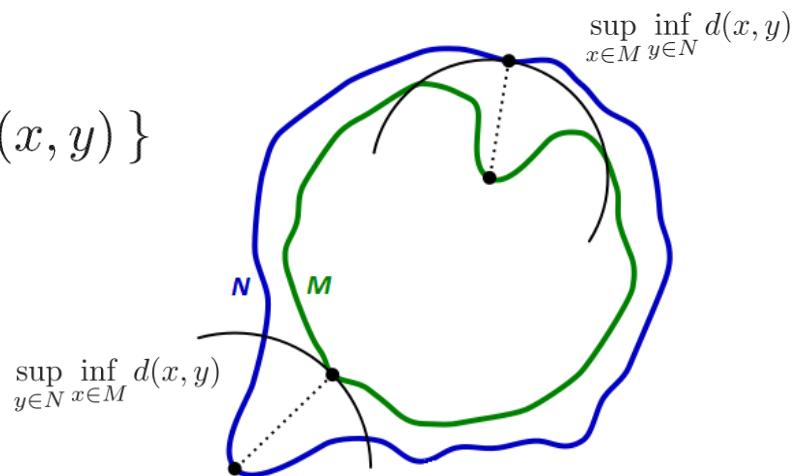


$$d_Y(fx, fy) \leq d_X(x, y) \quad \forall x, y \in X$$

$\mathcal{H} : \text{CMS} \rightarrow \text{CMS}$

$\mathcal{H}(X, d) = \text{non-empty compact subsets of } X \text{ with Hausdorff metric } d^*$

$$d^*(M, N) = \max\left\{ \sup_{x \in M} \inf_{y \in N} d(x, y), \sup_{y \in N} \inf_{x \in M} d(x, y) \right\}$$



Accessability of the Hausdorff functor

Theorem.

van Breugel, Hermida, Makkai, Worrell (2007)

$\mathcal{H}(X, d) = \text{free metric join-semilattice on } (X, d)$

$$\mathbf{Jsl}(\mathbf{CMS}) \begin{array}{c} \xleftarrow{\perp} \\[-1ex] \xrightarrow{U} \end{array} \mathbf{CMS} \circlearrowleft \mathcal{H} \quad \Rightarrow \quad \mathcal{H} \text{ is accessible}$$

Makkai & Pare (1989)

preserves λ -filtered colimits for **some** λ

\mathcal{A} category with $\times \dots$ $\mathbf{Jsl}(\mathcal{A}) :$ $A \times A \xrightarrow{\alpha} A$ such that

$$\begin{array}{ccc} A \times A \times A & \xrightarrow{\alpha \times \text{id}} & A \times A \\ \downarrow \text{id} \times \alpha & & \downarrow \alpha \\ A \times A & \xrightarrow{\alpha} & A \end{array}$$

associative

$$\begin{array}{ccc} A \times A & & \\ \text{swap} \downarrow & \searrow \alpha & \\ A \times A & \xrightarrow{\alpha} & A \end{array}$$

commutative

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \times A \\ & \swarrow \parallel & \downarrow \alpha \\ & A & \end{array}$$

idempotent

Accessability of the Hausdorff functor

Theorem.

van Breugel, Hermida, Makkai, Worrell (2007)

$\mathcal{H}(X, d) = \text{free metric join-semilattice on } (X, d)$

$$\text{Jsl}(\text{CMS}) \begin{array}{c} \xleftarrow{\perp} \\[-1ex] \xrightarrow{U} \end{array} \text{CMS} \circlearrowleft \mathcal{H}$$

Makkai & Pare (1989)

Can we do better?

preserves accessibly

for some λ

\mathcal{A} category with $\times \dots$

$\text{Jsl}(\mathcal{A}) : A \times A \xrightarrow{\alpha} A$

such that

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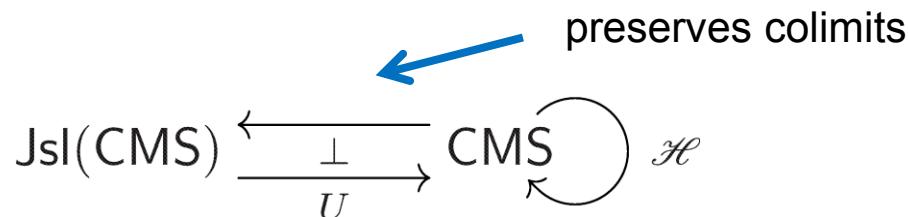
$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \times A \\ & \swarrow & \downarrow \alpha \\ & & A \end{array}$$

idempotent

Yes, we can!

$$\text{Jsl}(\text{CMS}) \xrightleftharpoons[\quad U \quad]{\perp} \text{CMS} \circlearrowleft \mathcal{H}$$

preserves colimits



Proposition. \mathcal{A} lfp category \mathcal{T} algebraic theory

Then $\text{Alg}_{\mathcal{A}}\mathcal{T}$ lfp and $U^{\mathcal{T}}: \text{Alg}_{\mathcal{A}}\mathcal{T} \rightarrow \mathcal{A}$ finitary

\mathcal{T} = theory of semilattices \dots $\text{Alg}_{\text{CMS}}\mathcal{T} = \text{Jsl}(\text{CMS})$

Bad news. CMS is not lfp!

But: CMS is locally **countably** presentable

Proposition holds for locally countably presentable categories

$\implies U: \text{Jsl}(\text{CMS}) \rightarrow \text{CMS}$ is countably accessible

$\implies \mathcal{H}: \text{CMS} \rightarrow \text{CMS}$ is countably accessible

Yes, we can!

$$\text{Jsl}(\text{CMS}) \xrightleftharpoons[\quad U \quad]{\perp} \text{CMS} \circlearrowleft \mathcal{H}$$

preserves colimits

Proposition. \mathcal{A} lfp category \mathcal{T} algebraic theory

Then $\text{Alg}_{\mathcal{A}}\mathcal{T}$ lfp and $U^{\mathcal{T}}: \text{Alg}_{\mathcal{A}}\mathcal{T} \rightarrow \mathcal{A}$ finitary

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Bad news. CMS is not lfp!

But: CMS is locally **countably** presentable
Proposition holds for countably presentable categories

$\Rightarrow U: \text{Jsl}(\text{CMS}) \rightarrow \text{CMS}$ is countably accessible

\Rightarrow **One can still do better!** $\text{Jsl}(\text{CMS}) \rightarrow \text{CMS}$ is countably accessible

Finitaryness of the Hausdorff functor

Theorem. $\mathcal{H}: \text{CMS} \rightarrow \text{CMS}$ preserves filtered colimits.

Proof.

1. $\text{CMS} \underset{\perp}{\underset{\curvearrowleft}{\longleftrightarrow}} \text{PMS}$ reflective subcategory

pseudometric spaces

2. $U^{\mathcal{T}}: \text{Alg}_{\text{PMS}} \mathcal{T} \rightarrow \text{PMS}$ finitary

3. $U': \text{Jsl}(\text{PMS}) \rightarrow \text{PMS}$ finitary $\implies U: \text{Jsl}(\text{CMS}) \rightarrow \text{CMS}$ finitary

category-theoretic argument

Presentation of the Hausdorff functor

The same as for the non-empty finite power-set functor $\mathcal{P}_f^{\neq\emptyset}: \text{Set} \rightarrow \text{Set}$.

separable spaces
= countably presentable $\longrightarrow \mathcal{C} \hookrightarrow \text{CMS} \longleftarrow$ locally countably presentable

signature $\Sigma_n = \begin{cases} \{\sigma_n\} & \text{if } n \text{ non-empty finite discrete} \\ \emptyset & \text{else} \end{cases}$

$$H_\Sigma X = \coprod_{0 < n < \omega} X^n = X^+$$

flat equations $1 \xrightarrow[u]{u'} H_\Sigma k$ $\sigma_\ell(x_0, \dots, x_{\ell-1}) = \sigma_k(y_0, \dots, y_{k-1})$
iff $\{x_0, \dots, x_\ell\} = \{y_0, \dots, y_k\}$

Proposition. The joint coequalizer of all u, u' is

$$\varepsilon: H_\Sigma \rightarrow \mathcal{H} \quad (\sigma_n, x_0, \dots, x_{n-1}) \mapsto \{x_0, \dots, x_{n-1}\}$$

Proof. 1. ε_X non-expanding

2. ε_X coequalizer: $1 \xrightarrow[u]{u'} H_\Sigma k \longrightarrow \mathcal{P}_f^{\neq\emptyset} X \xrightarrow{\text{dense}} \mathcal{H} X$

Conclusions and future work

Finitary functors on Ifp categories are precisely those having a finitary presentation

The Hausdorff functor is finitary and has a presentation by operations with finite arity

Future work

Kantorovich functor on CMS (for modelling probabilistic non-determinism)

Relation of our presentations to rank-1 presentations as in Bonsangue & Kurz