UNIVERSITY OF Southampton

School of Electronics and Computer Science

RELATIONAL PRESHEAVES AS LABELLED TRANSITION SYSTEMS

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CMCS `12, Tallinn, 01/04/12

MODELS FOR CONCURRENCY

 

(Winskel and Nielsen `93)

- **Category Theory ought *to inform* Concurrency Theory**
  - characterise (or improve) common constructions using universal properties (e.g. limits, colimits, adjoints)
  - use universal properties to identify interesting constructions and get a quick "feel" for any particular model

PLAN

- **Examples of algebraic structure on labels**
  - Ex1: observability, weak bisimulation, tau-closure
  - Ex2: operational accounts of wiring
  - Ex3: Jensen's weak reactive systems, tile systems
- LTSs categorically
- Introduction to relational presheaves
- Adjoints to change of base

EXAMPLE 1: OBSERVABILITY & WEAK BISIMULATION

- CCS, pi, ... have special transitions with tau-labels
- tau-labelled transitions are normally considered to be “silent” so unobservable
- equivalences should not distinguish systems that differ only by tau actions
  - one possibility: change the definition of bisimilarity to Milner’s weak bisimilarity

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- one possibility: change the definition of bisimilarity to Milner's weak bisimilarity

This is anathema in my religion: the Monotheistic Church of Bisimilarity

WEAK BISIMULATION

Another way: close LTS with the following two rules and consider bisimilarity

$$\frac{}{P \xrightarrow{\tau} P} \qquad \frac{P \xrightarrow{\tau} Q \quad Q \xrightarrow{a} R \quad R \xrightarrow{\tau} S}{P \xrightarrow{a} S}$$

- isn't this kind of like making tau the identity of a monoid of actions?
- what is the mathematical status of this saturation?

EXAMPLE 2: OPERATIONAL THEORIES OF WIRING

(S ICE`09, S CONCUR`10; Bruni, Melgratti, Montanari CONCUR`11)

inspired by RFC Walters' work on Span(Graph)

- **Idea**: explore process calculi that have symmetric monoidal categories as their algebras of processes (terms up to bisimilarity)
  - real syntax (no structural congruence)
  - bisimilarity is a congruence wrt operations
  - extremely close operational correspondences with various variants of Petri nets with boundaries
  - interesting algebra of underlying (symmetric monoidal, compact closed, etc.) categories of processes

$$\frac{}{k \xrightarrow[0]{0} k} \qquad \frac{}{k \xrightarrow[0]{1} k{+}1} \qquad \frac{}{k{+}1 \xrightarrow[1]{0} k}$$

$$\frac{}{\mathsf{I} \xrightarrow[a]{a} \mathsf{I}} \qquad \frac{}{\mathsf{X} \xrightarrow[ba]{ab} \mathsf{X}}$$

$$\frac{}{\Delta \xrightarrow[aa]{a} \Delta} \qquad \frac{}{\bot \xrightarrow{a} \bot} \qquad \frac{}{\nabla \xrightarrow[a]{aa} \nabla} \qquad \frac{}{\top \xrightarrow[a]{} \top}$$

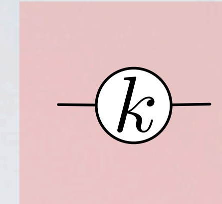
$$\frac{}{\wedge \xrightarrow[a(1-a)]{1} \wedge} \qquad \frac{}{\downarrow \xrightarrow{0} \downarrow} \qquad \frac{}{\vee \xrightarrow[1]{a(1-a)} \vee} \qquad \frac{}{\uparrow \xrightarrow[0]{} \uparrow}$$

$$\frac{}{\wedge \xrightarrow[00]{0} \wedge} \qquad \frac{}{\vee \xrightarrow[0]{00} \vee}$$

$$\frac{P \xrightarrow[\gamma]{\alpha} P' \quad Q \xrightarrow[\beta]{\gamma} Q'}{P;Q \xrightarrow[\beta]{\alpha} P';Q'} \qquad \frac{P \xrightarrow[\beta]{\alpha} P' \quad Q \xrightarrow[\delta]{\gamma} Q'}{P \otimes Q \xrightarrow[\beta\delta]{\alpha\gamma} P' \otimes Q'}$$

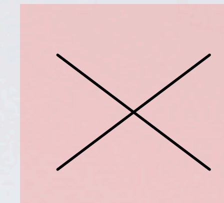
$$\frac{P \xrightarrow[\beta]{\alpha} P' \quad P' \xrightarrow[\beta']{\alpha'} Q}{P \xrightarrow[\beta+\beta']{\alpha+\alpha'} Q}$$

$$\dfrac{}{k \xrightarrow[0]{0} k} \qquad \dfrac{}{k \xrightarrow[0]{1} k+1} \qquad \dfrac{}{k+1 \xrightarrow[1]{0} k}$$

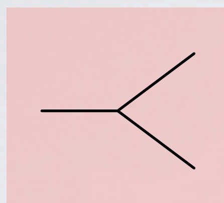


$$\dfrac{}{\mathsf{I} \xrightarrow[a]{a} \mathsf{I}}$$

$$\dfrac{}{\mathsf{X} \xrightarrow[ba]{ab} \mathsf{X}}$$

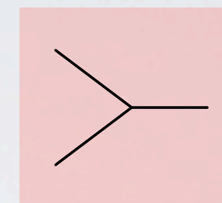


$$\dfrac{}{\Delta \xrightarrow[aa]{a} \Delta}$$

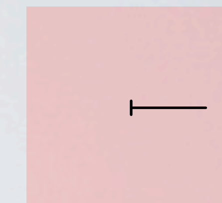


$$\dfrac{}{\perp \xrightarrow{a} \perp}$$

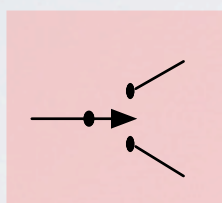
$$\dfrac{}{\nabla \xrightarrow[a]{aa} \nabla}$$



$$\dfrac{}{\top \xrightarrow[a]{} \top}$$

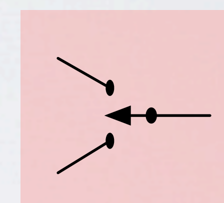


$$\dfrac{}{\Lambda \xrightarrow[a(1-a)]{1} \Lambda} \qquad \dfrac{}{\Lambda \xrightarrow[00]{0} \Lambda}$$



$$\dfrac{}{\downarrow \xrightarrow{0} \downarrow}$$

$$\dfrac{}{\mathsf{V} \xrightarrow[1]{a(1-a)} \mathsf{V}} \qquad \dfrac{}{\mathsf{V} \xrightarrow[0]{00} \mathsf{V}}$$



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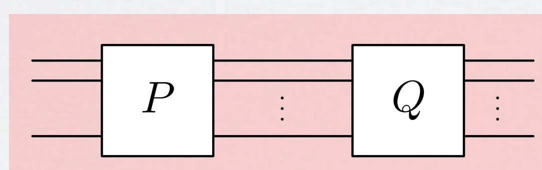

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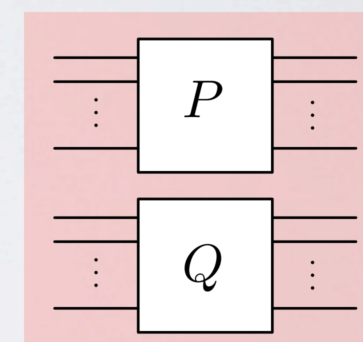

$$\dfrac{P \xrightarrow[\beta]{\alpha} P' \qquad P' \xrightarrow[\beta']{\alpha'} Q}{P \xrightarrow[\beta+\beta']{\alpha+\alpha'} Q}$$

UNEXPECTED BEHAVIOUR

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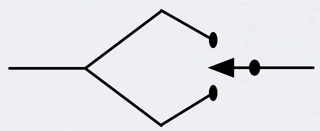
$\Delta \; ; \; \mathsf{V}$ 

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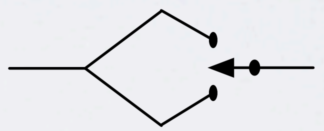
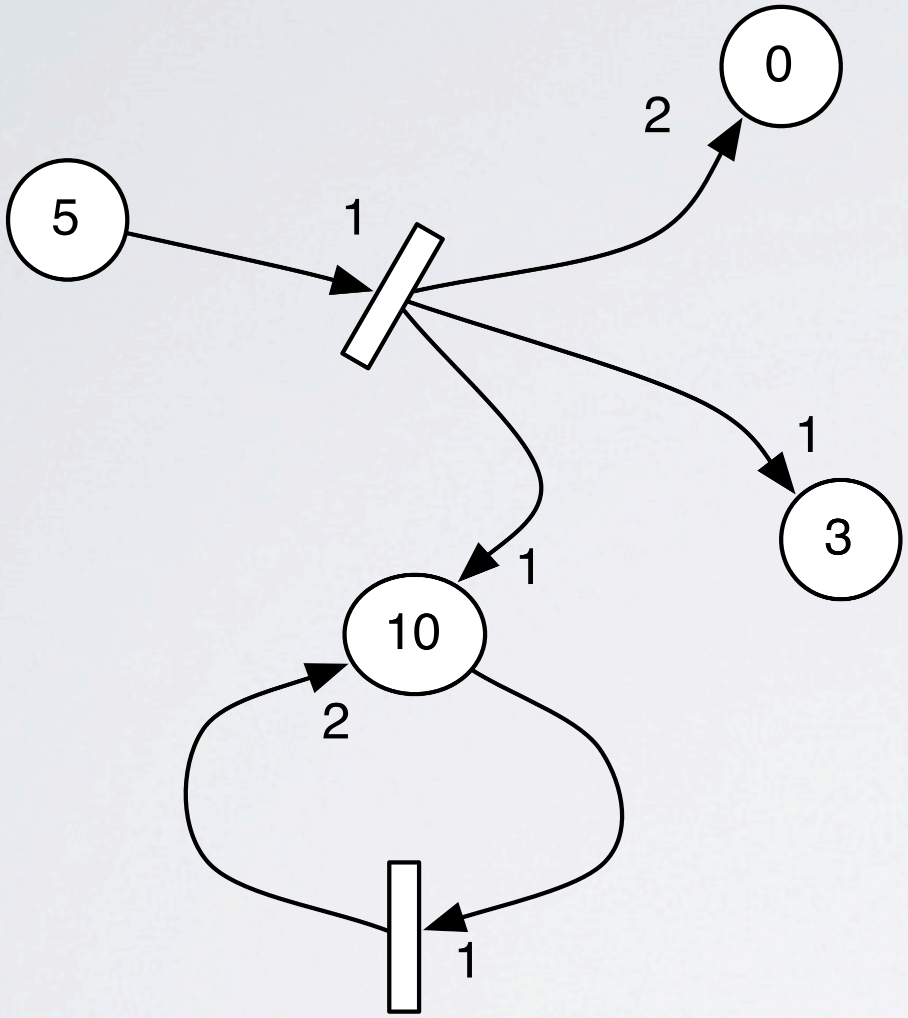
$\Delta \,;\, \mathsf{V}$ 

$$\frac{P \xrightarrow[\beta]{\alpha} P' \qquad P' \xrightarrow[\beta']{\alpha'} Q}{P \xrightarrow[\beta+\beta']{\alpha+\alpha'} Q}$$

$$\frac{\dfrac{}{\Delta \xrightarrow[11]{1} \Delta} \qquad \dfrac{\mathsf{V} \xrightarrow[01]{1} \mathsf{V} \quad \mathsf{V} \xrightarrow[10]{1} \mathsf{V}}{\mathsf{V} \xrightarrow[11]{2} \mathsf{V}}}{\Delta \,;\, \mathsf{V} \xrightarrow[2]{1} \Delta \,;\, \mathsf{V}}$$

P/T NETS



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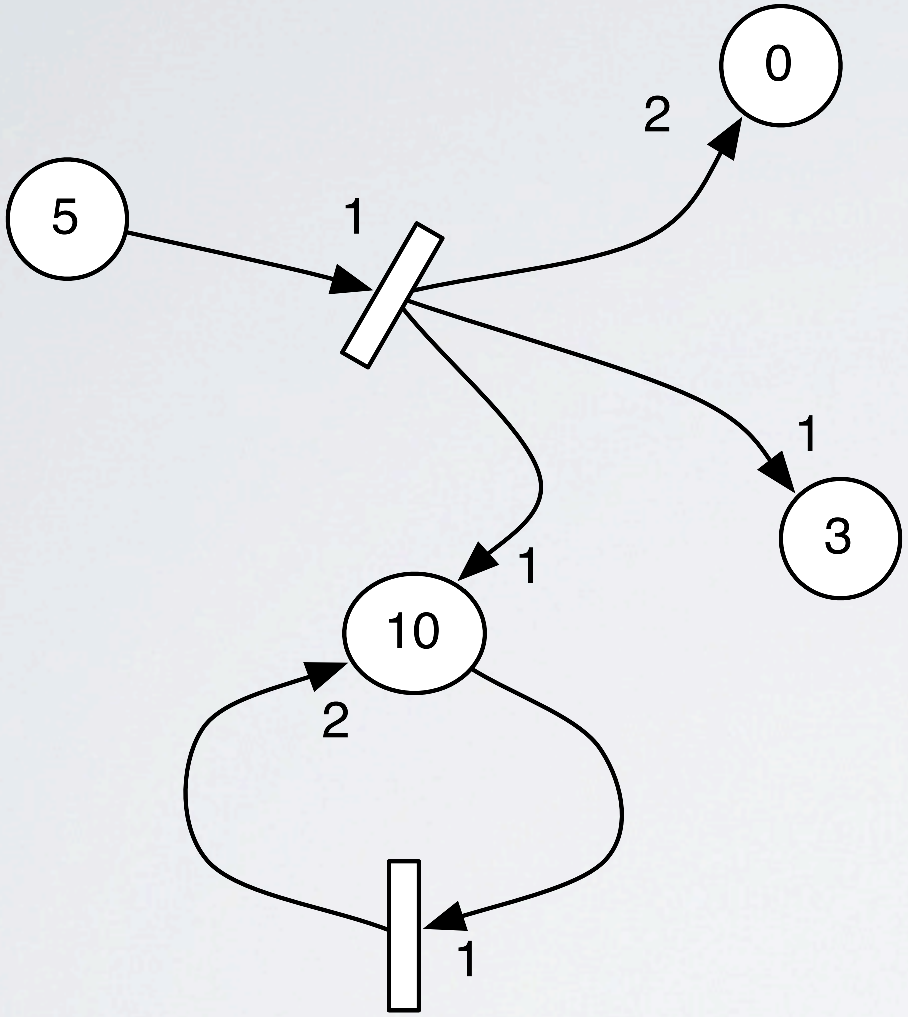

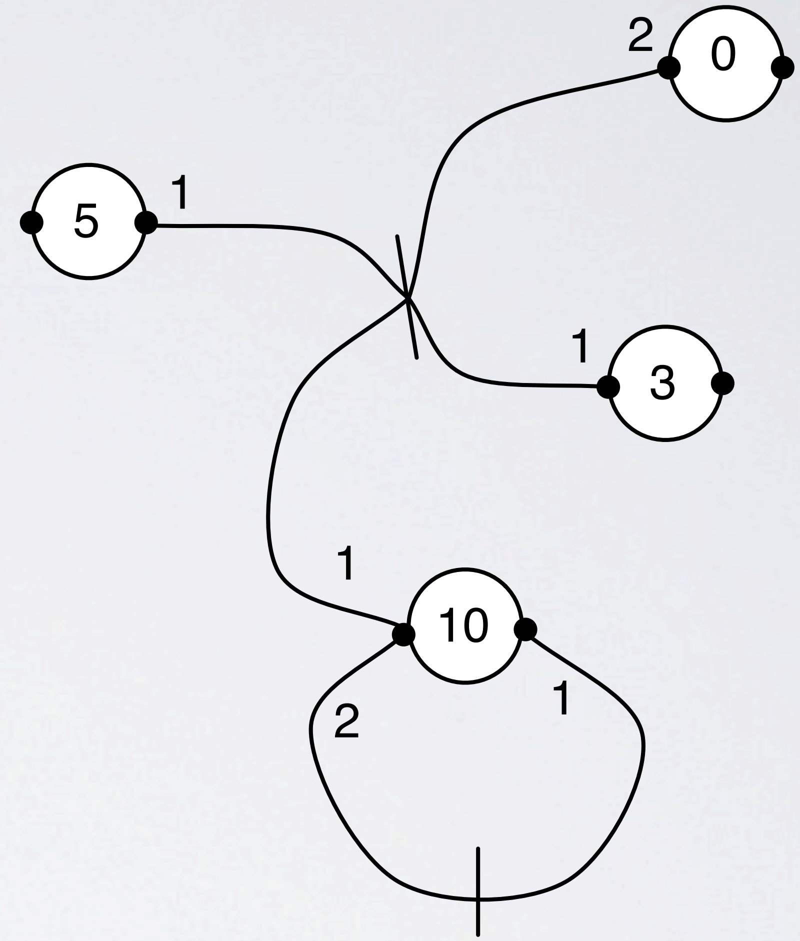

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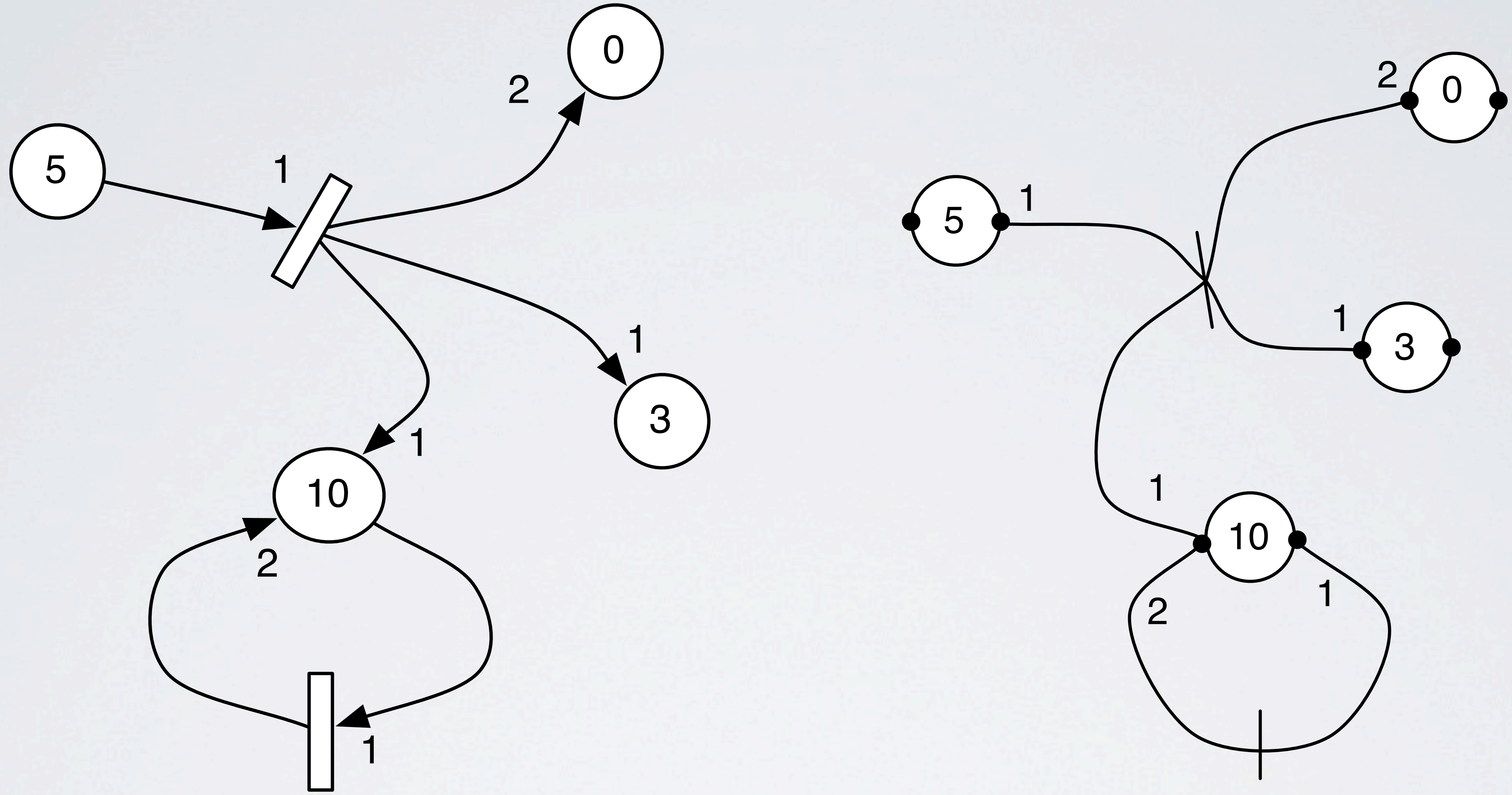


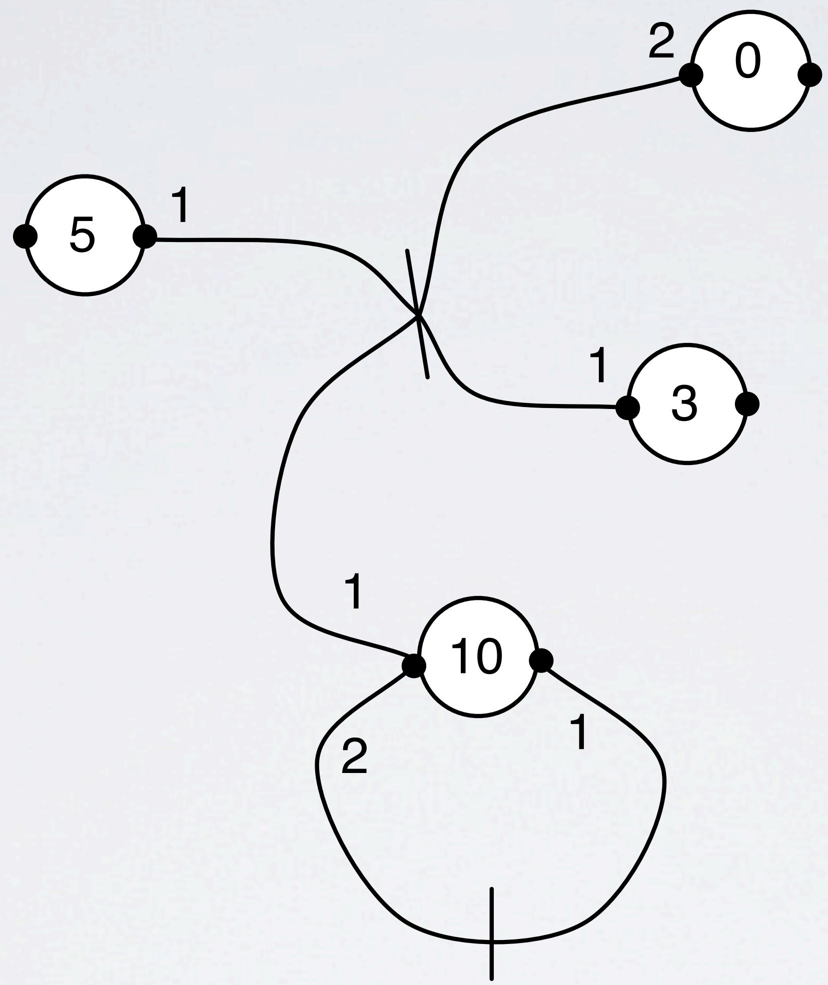
<u>semantics</u>:

states - multisets of places (markings)

transitions - **X** → **Y** for if M a multiset of transitions

$$\mathbf{X} + \mathrm{post}(\mathbf{M}) = \mathbf{Y} + \mathrm{pre}(\mathbf{M})$$

P/T NETS WITH BOUNDARIES



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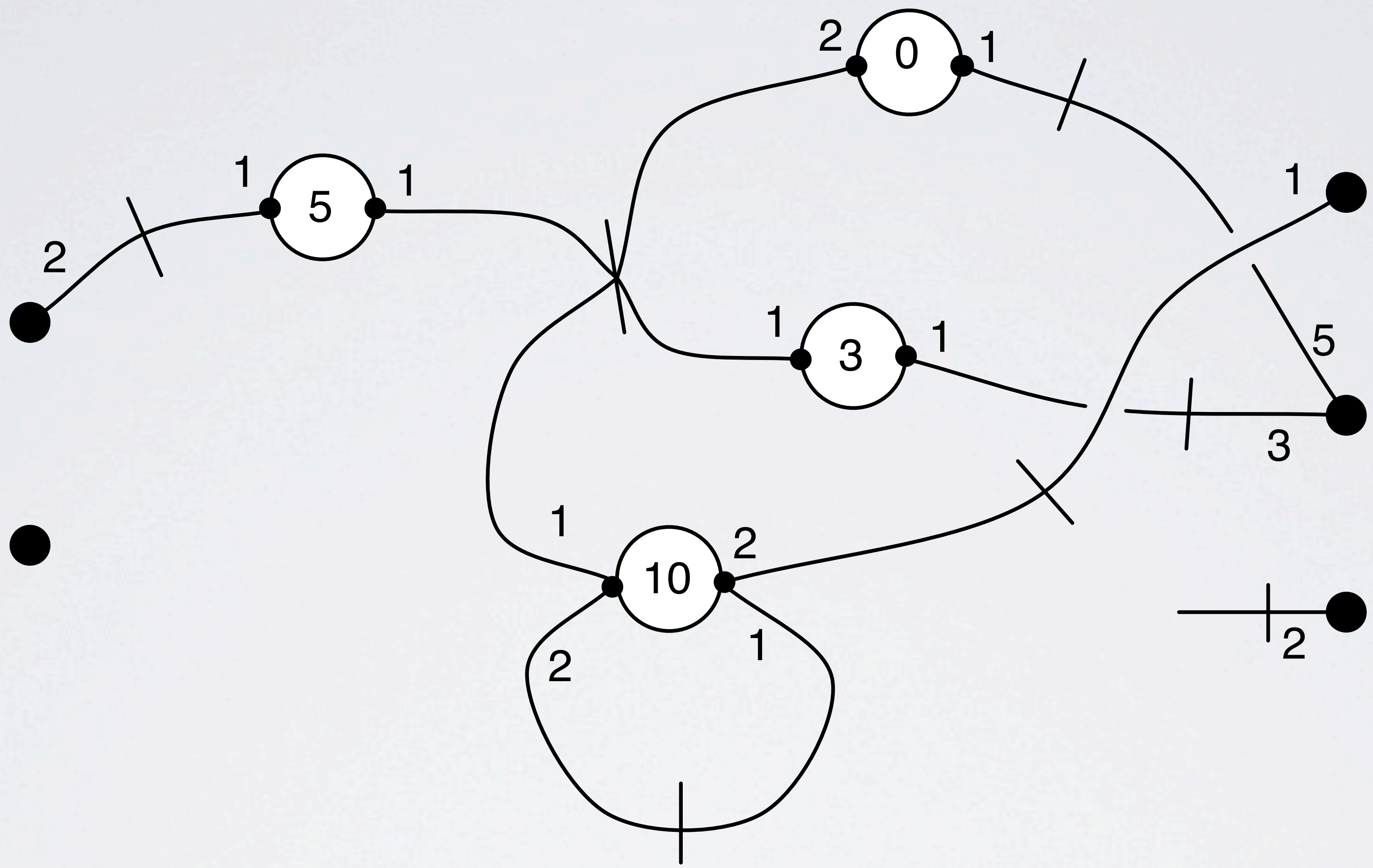

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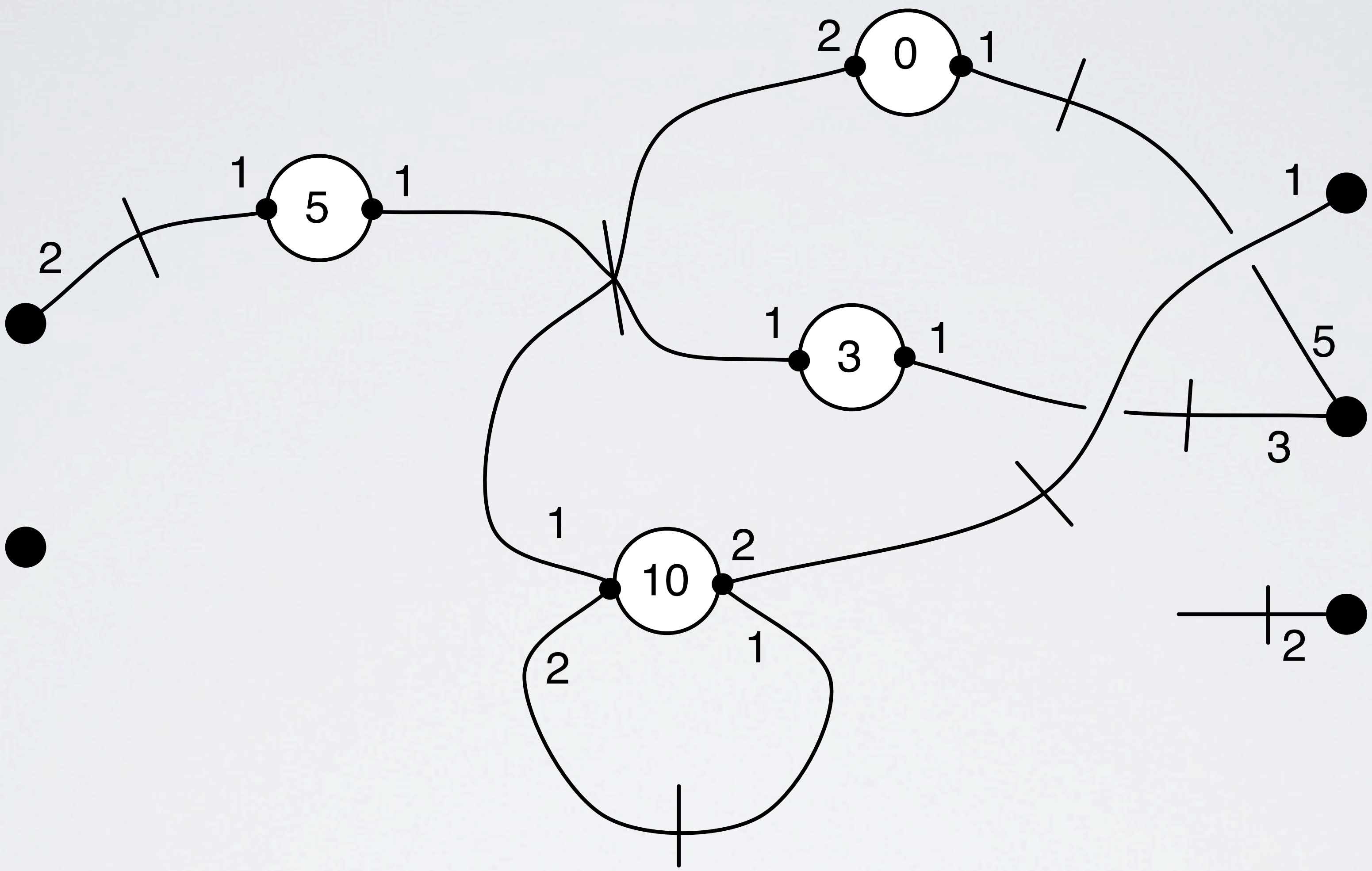


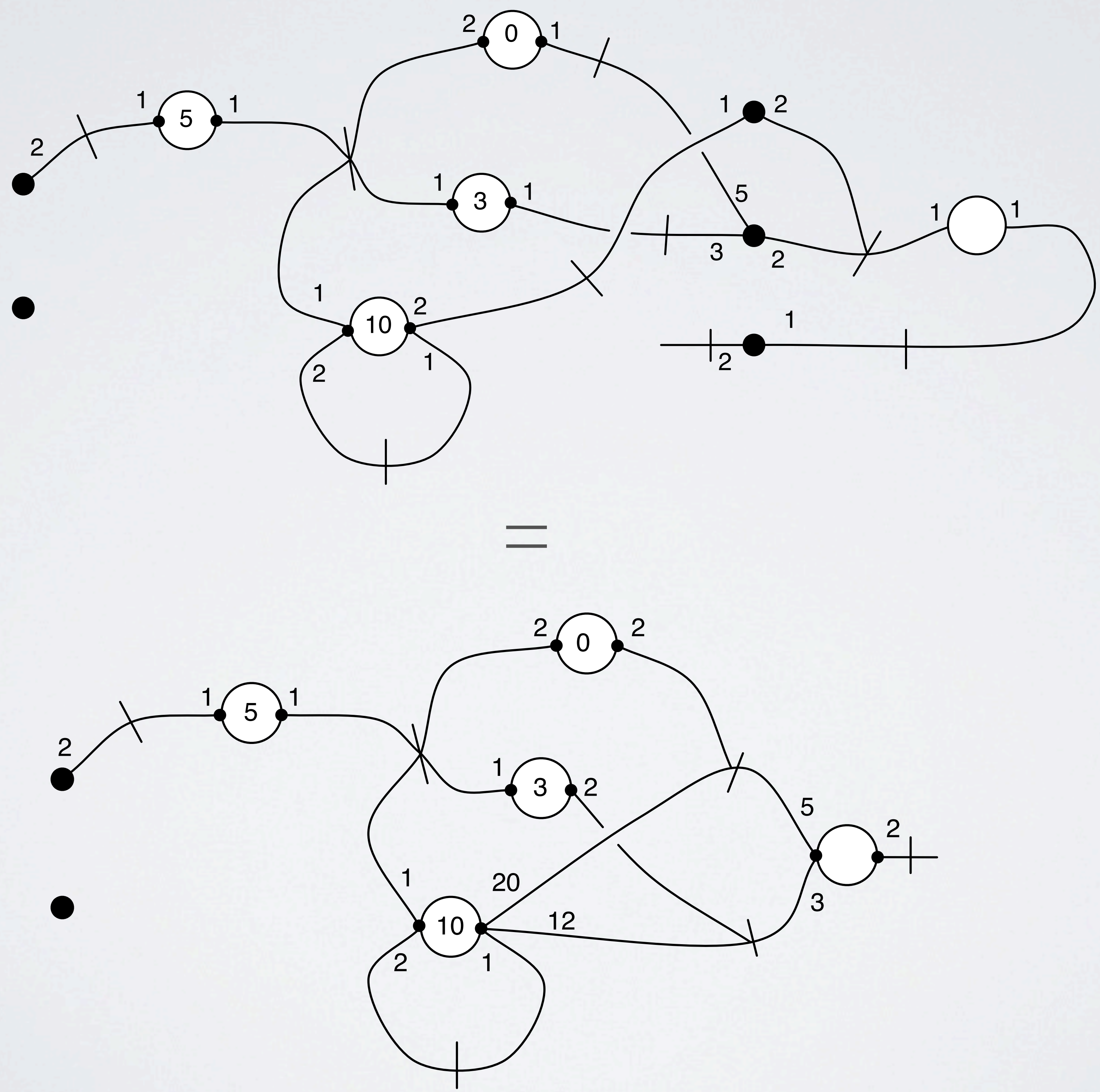
semantics:

states - multisets of places (markings)

transitions - $\mathbf{X} \xrightarrow[\mathbf{b}]{\mathbf{a}} \mathbf{Y}$ for if **M** a multiset of transitions

$\mathbf{X} \rightarrow \mathbf{Y}$ & $\mathbf{a}$ = source(**M**), $\mathbf{b}$ = target(**M**)

COMPOSITION



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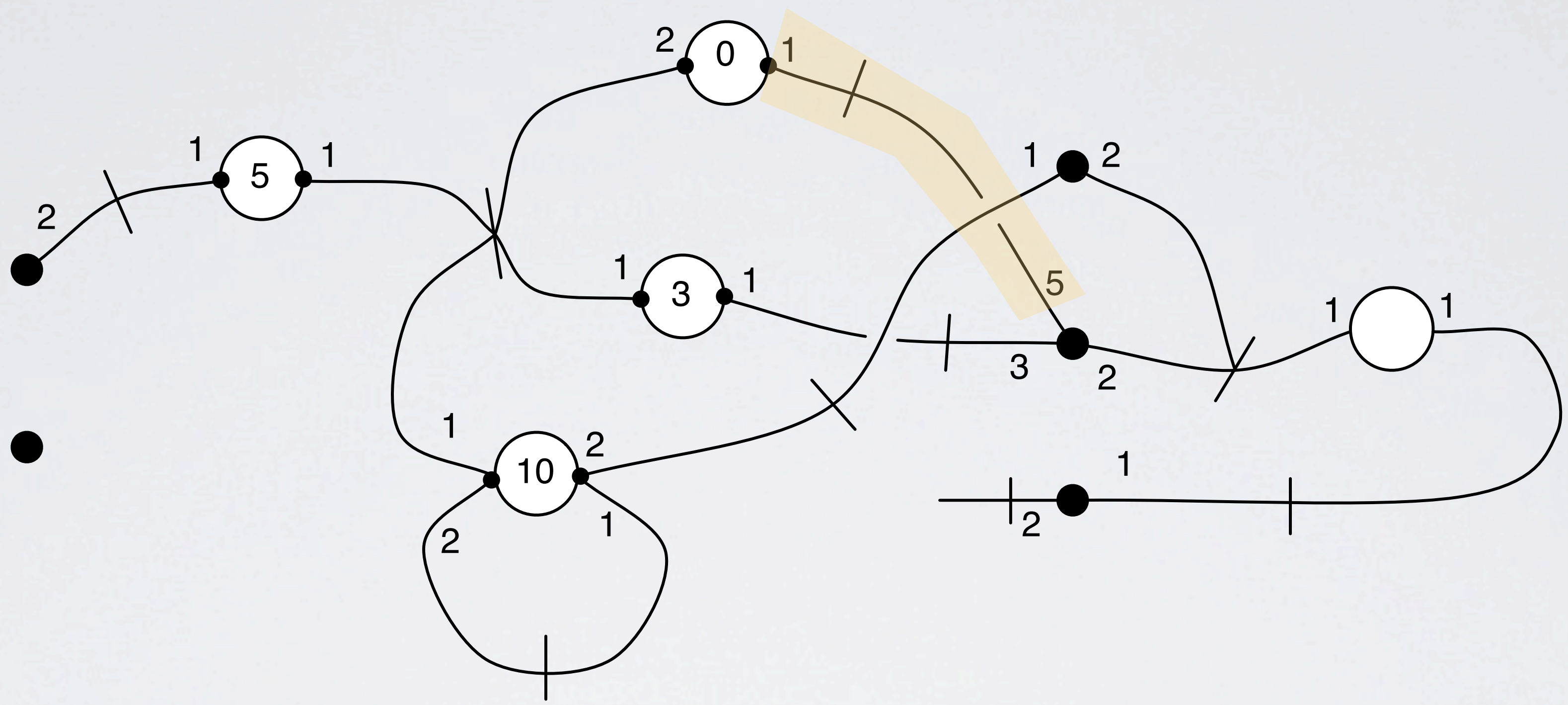

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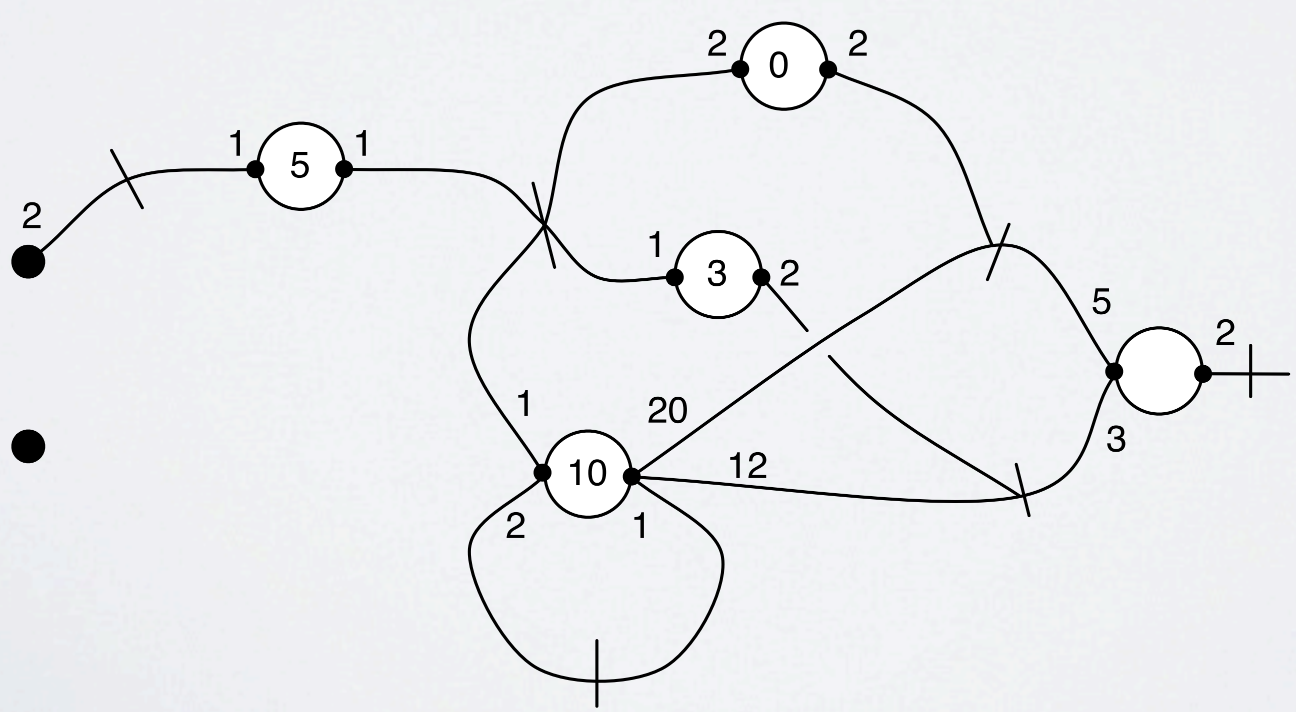

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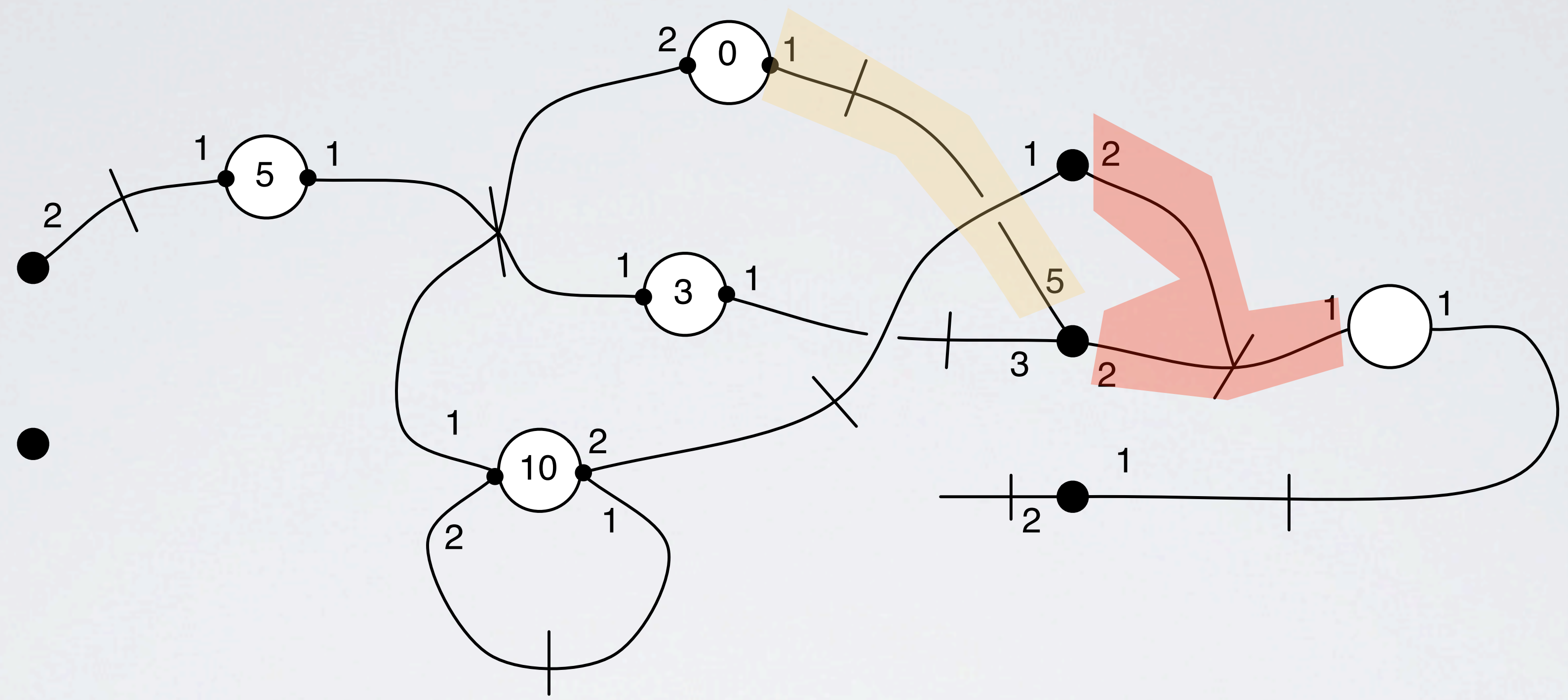

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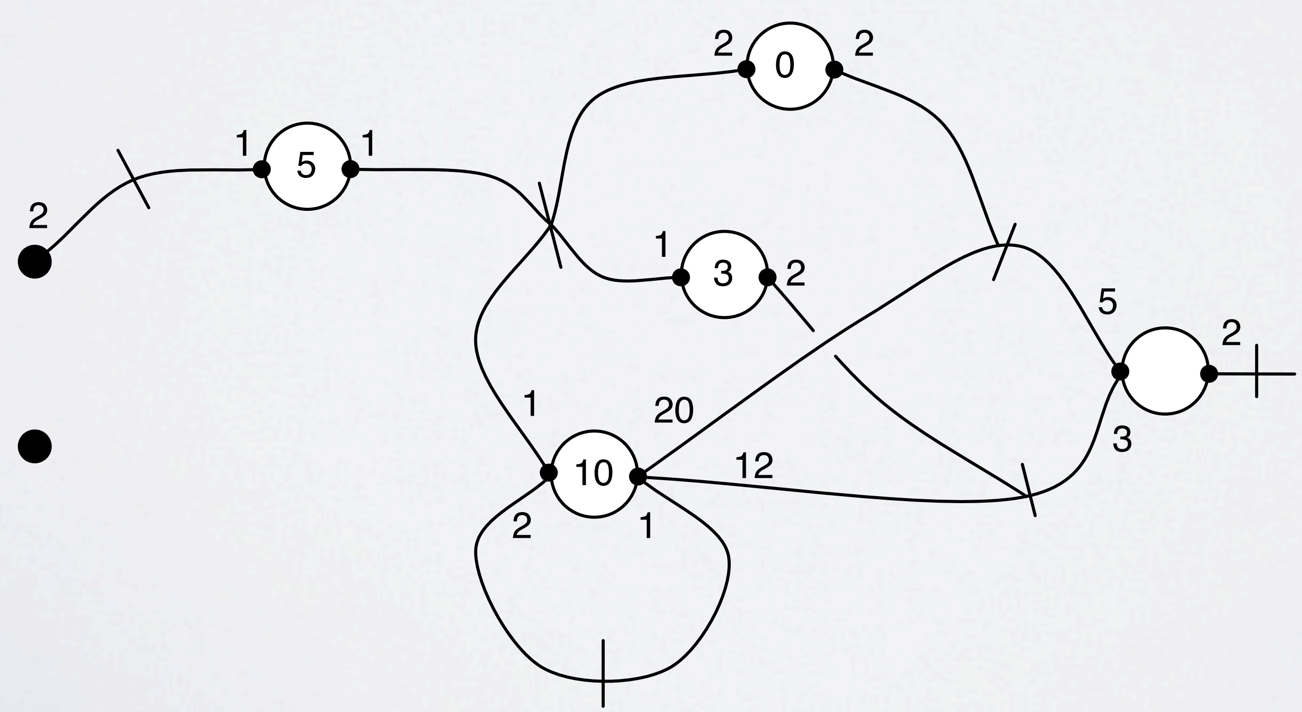

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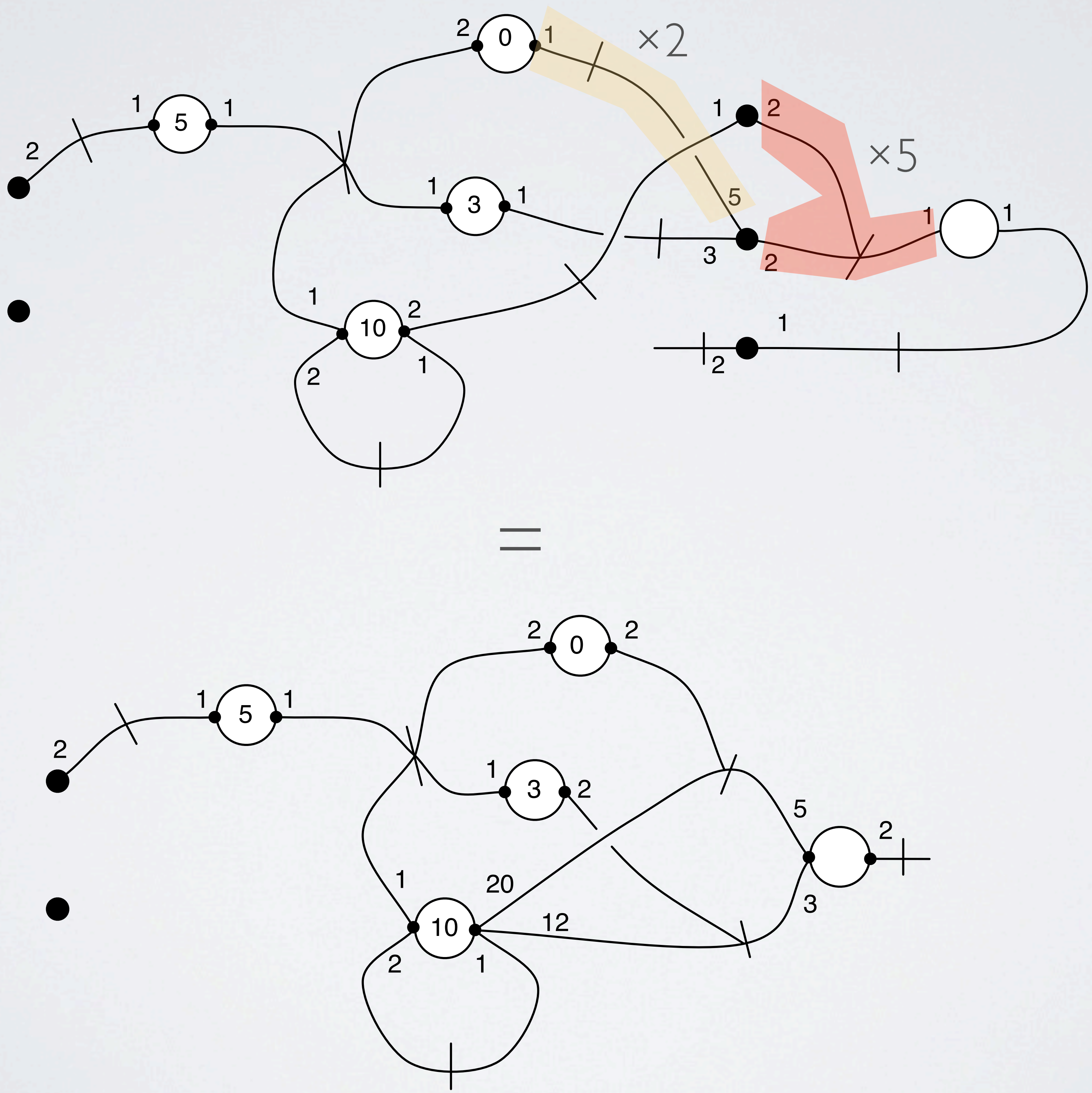

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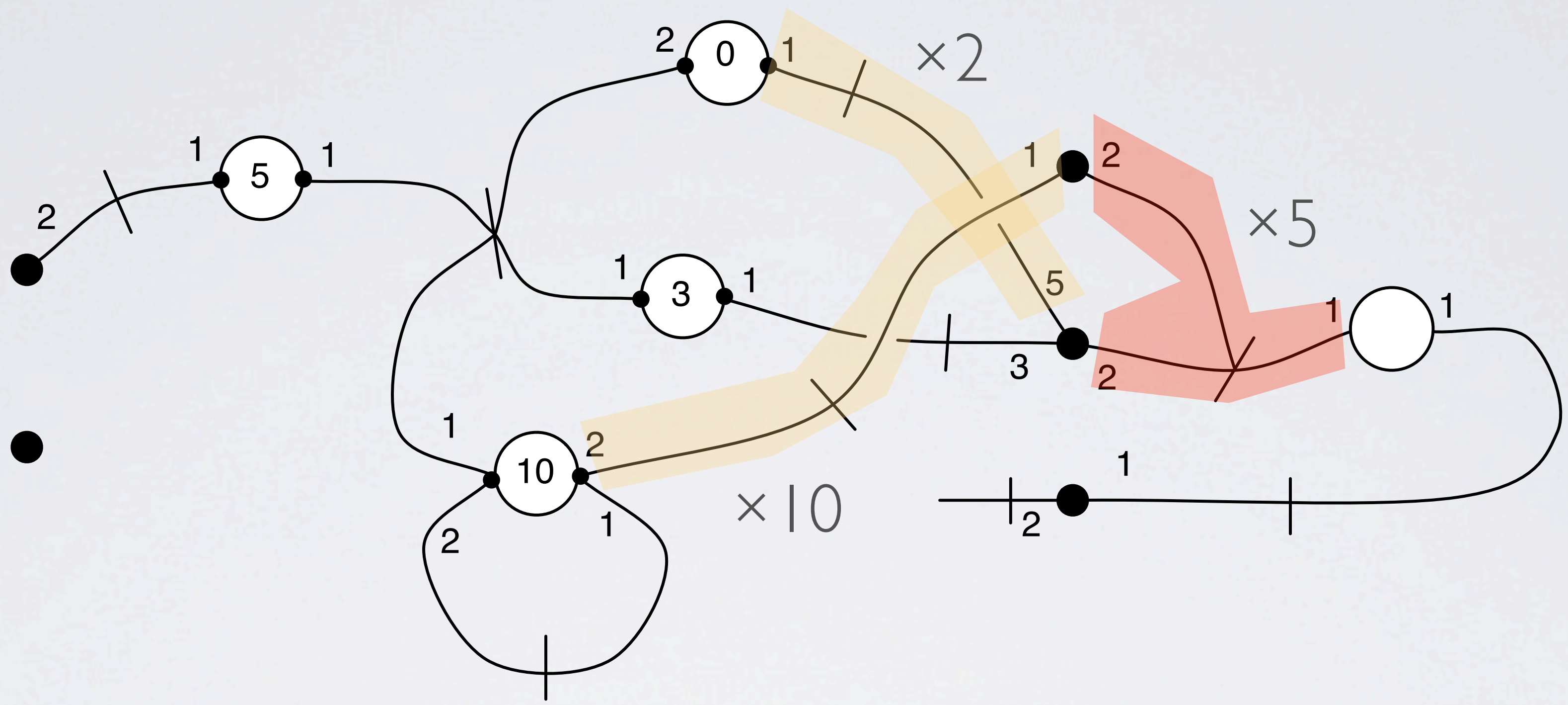

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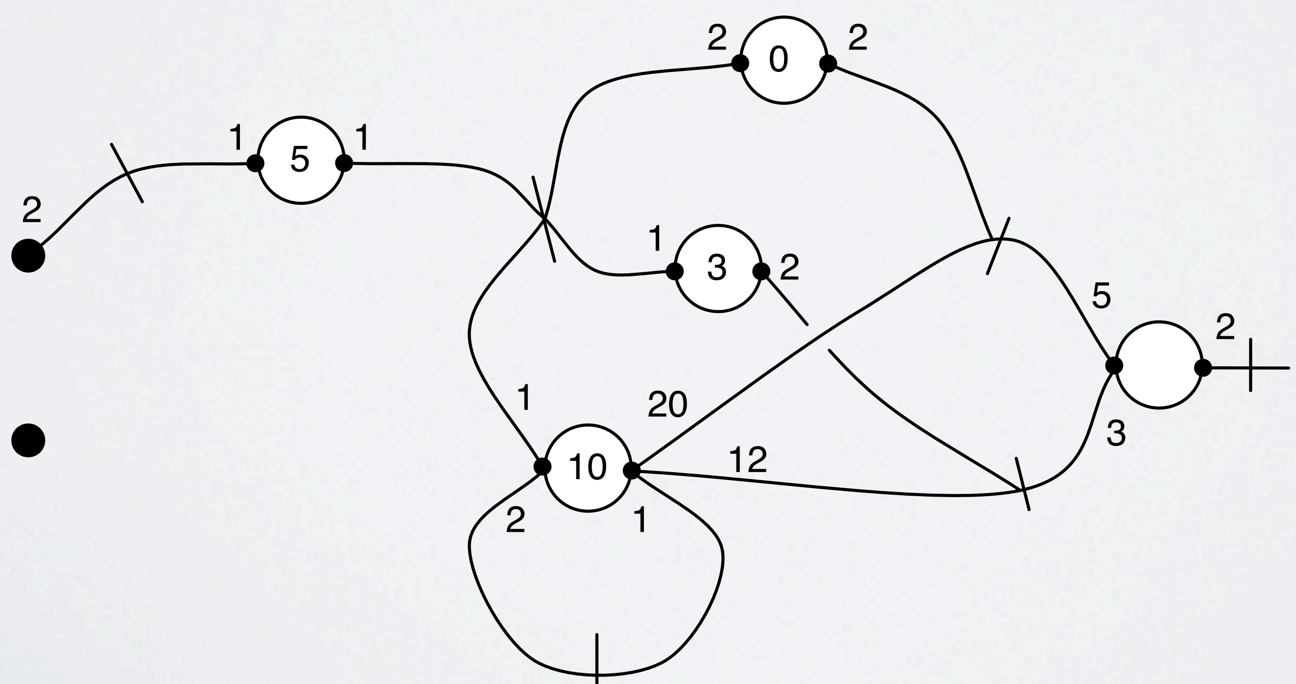

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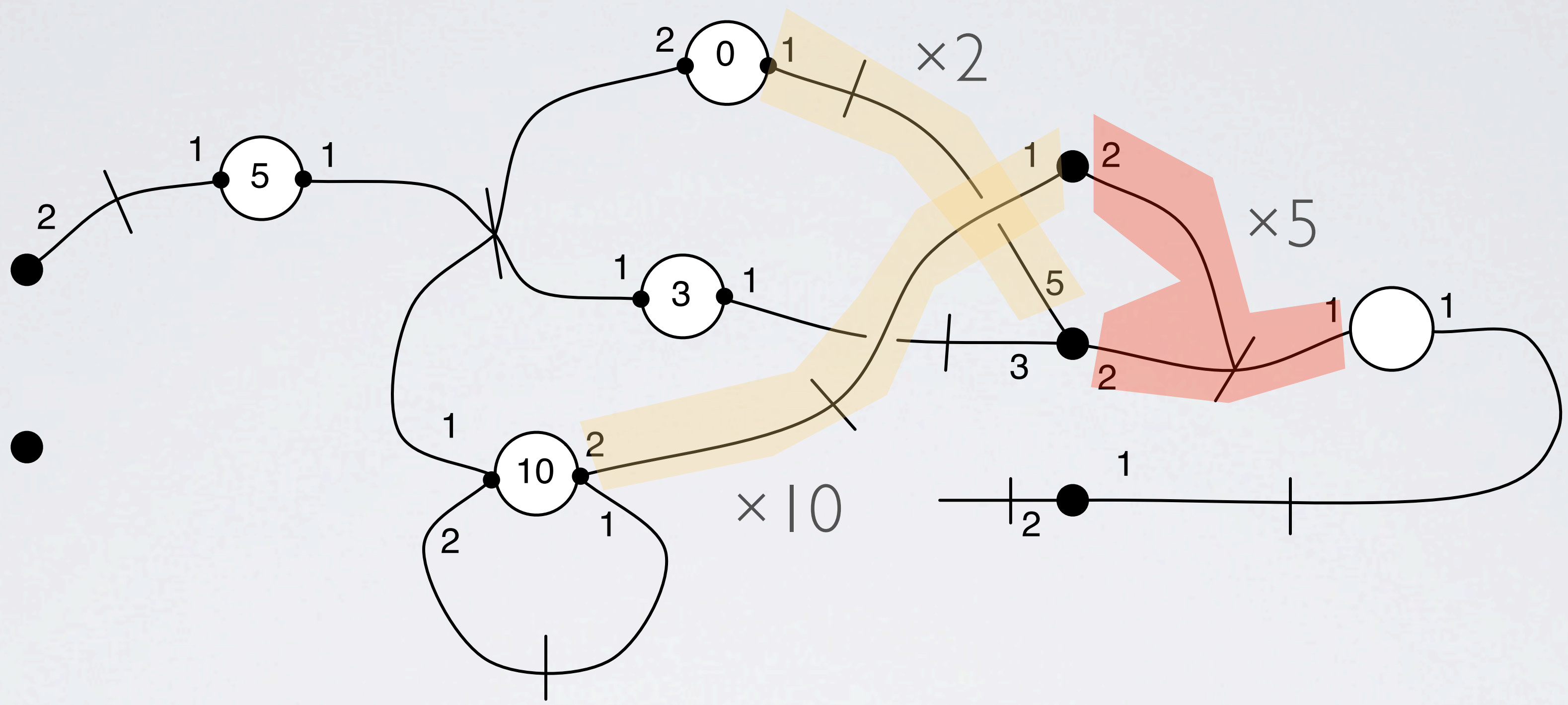

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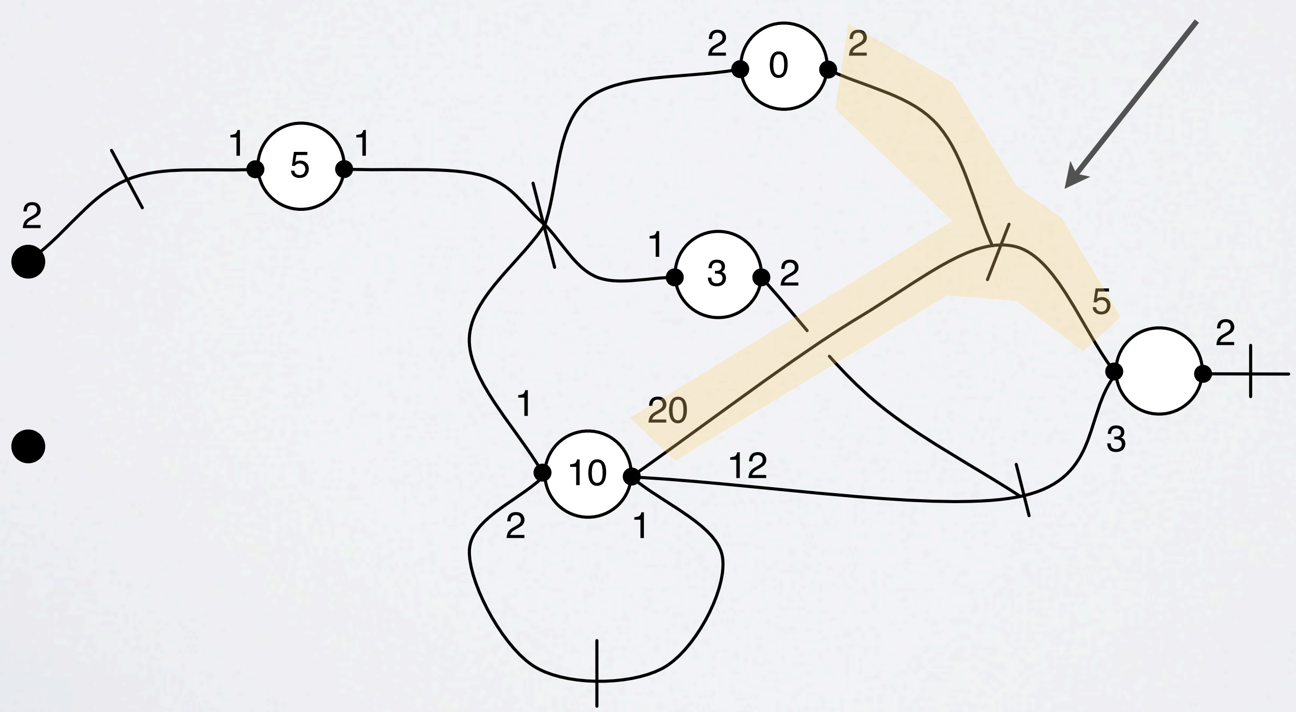
minimal synchronisation



CORRESPONDENCE

- for each net N there is a term t_N such that { (N, t_N) | N a P/T net with boundaries} is a bisimulation
- there is an recursively defined translation from terms to nets such that { (t, $[N\_t]_{\cong}$) | is a bisimulation }

- induces an isomorphism of underlying process categories

EXAMPLE 3: OLE JENSEN'S WEAK EQUIVALENCE FOR REACTIVE SYSTEMS

Definition 3.18 (weak reaction) We say that reaction rules (p, p') and (q, q') are *compatible* if p' and q are consistent and equations (1) and (2) above hold.

For compatible rules (p, p') and (q, q') we their *composition* $(p, p') \cdot (q, q')$ is defined as the rule $(P \circ p, Q' \circ q')$, where (P, Q') is an IPO of (p', q).

We call a rule of the form (id_I, id_I) an *identity rule*.

For $\mathcal{R}$ a set of rules we define its *weakening*, written $\mathcal{W}(\mathcal{R})$, as the result of adding all identity rules and then closing under composition of compatible rules. We extend $\mathcal{W}$ to a functor that sends a reactive system $(\mathbf{C}, \mathcal{R})$ to $(\mathbf{C}, \mathcal{W}(\mathcal{R}))$.

Define the *weak reaction relation* $\Rightarrow$ in $\mathbf{C}$ as the reflection of reaction in $\mathcal{W}(\mathbf{C})$; that is, $a \Rightarrow a'$ in $\mathbf{C}$ iff $a \rightarrow a'$ in $\mathcal{W}(\mathbf{C})$. ■

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Lemma 3.22 In a reactive system with all RPOs the following hold:

(1) $a \overset{id}{\Longrightarrow} a$.

(2) If $a \xrightarrow{L} a'$ then $a \overset{L}{\Longrightarrow} a'$;

(3) If $a \overset{L_1}{\Longrightarrow} \cdots \overset{L_n}{\Longrightarrow} a'$ and $L = L_n \circ \cdots \circ L_1$ then $a \overset{L}{\Longrightarrow} a'$;

(4) If $a \overset{L}{\Longrightarrow} a'$ then $a \xrightarrow{L_1} \cdots \xrightarrow{L_n} a'$ for some $L_1, \cdots, L_n$ such that $L = L_n \circ \cdots \circ L_1$.

LTS++

- Several LTSs have algebraic structure on the set of labels
  - labels are elements of a monoid
  - transitions are closed under:

$$\frac{}{P \xrightarrow{\iota} P} \qquad \frac{P \xrightarrow{a} Q \quad Q \xrightarrow{b} R}{P \xrightarrow{a \star b} R}$$

  - labels are arrows of a category (e.g. reactive systems), transitions are closed under identities and composition
  - other examples: Span(Graph), tile systems, ...

PLAN

- Examples of algebraic structure on labels
- **LTSs categorically**
  - presheaves and open maps
  - coalgebra
  - how to capture algebraic structure on labels?
- Introduction to relational presheaves
- Adjoints to change of base

PRESHEAVES

Winskel Nielsen `96 - Presheaves as transition systems

- presheaves can be thought of as transition systems via an elements construction
- indexing category can be thought of as a "category of paths"
- morphisms are functional simulations
- functional bisimulations are characterised as open maps wrt path category obtained via Yoneda

COALGEBRA

- LTS are coalgebras for the $P(A\times -)$ functor on **Set**
  - the labels are "wrapped up" inside the functor
  - coalgebra morphisms are functional bisimulations
  - ...
- Coalgebras are versatile with a mature theory and 100s of great papers
  - some constructions are notoriously tricky (e.g. weak bisimulation)

LABELS WITH STRUCTURE FROM COALGEBRA TO?

$$\dfrac{X \to \mathcal{P}(A \times X)}{\dfrac{X \to \mathcal{P}(X)^A}{A \to \mathcal{P}(X)^X}}$$

LABELS WITH STRUCTURE FROM COALGEBRA TO?

$$\dfrac{X \to \mathcal{P}(A \times X)}{\dfrac{X \to \mathcal{P}(X)^A}{A \to \mathcal{P}(X)^X}}$$

some kind of
structure preserving
$A \to \mathbf{Rel}$
??

PLAN

- Examples of algebraic structure on labels
- LTSs categorically
- **Introduction to relational presheaves**
  - relational presheaves and their morphisms
    - functional morphisms vs relational morphisms
  - quantaloids
  - examples, correspondence with simulations and bisimulations
- Adjoints to change of base

RELATIONAL PRESHEAVES

(Kimmo Rosenthal, The theory of quantaloids)

Relational presheaf **C** on = lax functor from **C**^{op} to **Rel**

$$h\colon \mathbf{C}^{op} \to \mathbf{Rel}$$

for each object *h* gives a set,
for each arrow a relation

laxness means:

$$h(b); h(a) \subseteq h(a; b) \qquad I_{h(x)} \subseteq h(I_x)$$

RELATIONAL PRESHEAVES

- AKA:
  - specification structures (Abramsky, Gay, Nagarajan)
    - generalised type theories
    - **C** = **Set**, **Rel**, category of domains, particular choices of functors
  - relational variable sets (Ghilardi and Meloni)
    - models for propositional logic
  - dynamic sets (Stell), relsets (Niefield), ...

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in this talk **C** will be a monoid or
a category (monoidoloid?)

EXAMPLE

- Take **C** to be a 1-object category (monoid)

$$h\colon \mathbf{C}^{op} \to \mathbf{Rel}$$
$$* \longmapsto X$$
$$m \longmapsto X \nrightarrow X$$

+ lax functoriality

$$h(b); h(a) \subseteq h(a; b) \qquad I_{h(x)} \subseteq h(I_x)$$

so same thing as a transition system, with labels in M satisfying

$$\frac{}{x \xrightarrow{\iota} x} \qquad \frac{x \xrightarrow{a} y \quad y \xrightarrow{b} z}{x \xrightarrow{a \star b} z}$$

MORPHISMS OF RELATIONAL PRESHEAVES

Functional morphisms

$$\mathcal{R}(\mathbf{C})$$

arrows - functional oplax natural transformations

$$\begin{tikzcd}
hC \arrow[r, "\varphi_C"] \arrow[d, "hf"', "|" marking] \arrow[dr, phantom, "\subseteq"] & h'C \arrow[d, "h'f", "|" marking] \\
hD \arrow[r, "\varphi_D"'] & h'D
\end{tikzcd}$$

Relational morphisms

$$\mathcal{R}^*(\mathbf{C})$$

arrows - oplax natural transformations

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hD \arrow[r, "\varphi_D"', "|" marking] & h'D
\end{tikzcd}$$

EXAMPLE

$\mathcal{R}^*(\mathbf{C})$

$$\begin{tikzcd} X \arrow[r, "\varphi", "|" marking] \arrow[d, "hm"', "-" marking] \arrow[dr, phantom, "\subseteq"] & X' \arrow[d, "h'm", "-" marking] \\ X \arrow[r, "\varphi"', "|" marking] & X' \end{tikzcd}$$

EXAMPLE

$\mathcal{R}^*(\mathbf{C})$

$$
\begin{tikzcd}
y & & & \\
& X \arrow[r, "\varphi", "|" marking] \arrow[d, "hm"', "|" marking] \arrow[dr, phantom, "\subseteq"] & X' \arrow[d, "h'm", "|" marking] & \\
x \arrow[uu, "m"] & X \arrow[r, "\varphi"', "|" marking] & X' &
\end{tikzcd}
$$

$x \qquad \varphi \qquad x'$

$\mathcal{R}^*(\mathbf{C})$

EXAMPLE

$$
\begin{tikzcd}
y & & \varphi & & y' \\
& X \arrow[rr, "\varphi", "|" marking] \arrow[d, "hm"', "|" marking] \arrow[drr, phantom, "\subseteq"] & & X' \arrow[d, "h'm", "|" marking] & \\
& X \arrow[rr, "\varphi"', "|" marking] & & X' & \\
x \arrow[uuu, "m"] & & \varphi & & x' \arrow[uuu, "m"']
\end{tikzcd}
$$

EXAMPLE

$\mathcal{R}^*(\mathbf{C})$

$$
\begin{tikzcd}
y & & \varphi & & y' \\
& X \arrow[rr, "\varphi"] \arrow[d, "hm"'] \arrow[drr, phantom, "\subseteq"] & & X' \arrow[d, "h'm"] & \\
& X \arrow[rr, "\varphi"'] & & X' & \\
x \arrow[uuu, "m"] & & \varphi & & x' \arrow[uuu, "m"']
\end{tikzcd}
$$

So functional simulations and ordinary simulations

RECAP

- For M a monoid we have

$\mathcal{R}(M)$	$\mathcal{R}^*(M)$
objects: labelled transition systems with monoidal structure on labels	
arrows: functional simulations	arrows: simulations 2-cells: inclusions

for general C, relational presheaves
can be considered LTSs via Grothendieck construction

QUANTALOIDS

- Are the categorification of quantales
  - quantale = complete lattice with monoidal structure that commutes with sup
  - quantaloid = locally small category in which homs are complete lattices and sups are preserved by composition in both directions
  - if $\mathbf{C}$ is locally small then $\mathcal{R}^*(\mathbf{C})$ is a quantaloid
    - in our examples this essentially means that unions of simulations are simulations

FUNCTIONAL BISIMULATIONS

- Are maps, i.e. left adjoints in the 2-categorical sense

$$h, h' \in \mathcal{R}^*(\mathbf{C}) \qquad \begin{array}{l} \varphi\colon h \to h' \\ \psi\colon h' \to h \end{array}$$

$I_h \subseteq \psi\varphi$	$\varphi\psi \subseteq I_{h'}$	
total	one-valued	i.e. phi is a function

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total	one-valued

i.e. phi is a function

$$x\varphi y \wedge y \xrightarrow{a} y' \Longrightarrow y\psi x \Longrightarrow$$
$$\exists x' .\, x \xrightarrow{a} x' \wedge y'\psi x' \Longrightarrow x'\varphi y'$$

ORDINARY LABELLED TRANSITION SYSTEMS

- Let A be a set of labels
- Labelled transition systems are exactly the ordinary functors

$$A^* \to \mathbf{Rel}$$

$LTS(A)$ = full subcategory of $\mathcal{R}^*(A^*)$

objects - labelled transition systems
arrows - simulations

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objects - labelled transition systems
arrows - simulations

Q. What are the other objects of $\mathcal{R}^*(A^*)$?

PLAN

- Examples of algebraic structure on labels
- LTSs categorically
- Introduction to relational presheaves
- **Adjoints to change of base**
  - Change of base
  - Niefield’s theorem
  - extensions and applications

CHANGE OF BASE

- Suppose we have a functor u : **D**→**C**

  change of base (2-)functors

  $$\mathcal{R}\mathbf{C} \xrightarrow{u^*} \mathcal{R}\mathbf{D} \qquad \mathcal{R}^*\mathbf{C} \xrightarrow{u^*} \mathcal{R}^*\mathbf{D} \qquad h \longmapsto h \circ u$$

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LTS Example:

Given a function f: A →B,

$$f^* : \mathcal{R}^*(B^*) \to \mathcal{R}^*(A^*)$$

takes an LTS with labels in A to LTS with same statespace and transitions

$$x \xrightarrow{a} x' \text{ in } f^*h \quad \Leftrightarrow \quad x \xrightarrow{fa} x' \text{ in } h$$

ADJOINTS TO CHANGE OF BASE IN $\mathcal{R}(\mathbf{C})$.

(Niefield - Change of base for relational variable sets 2004)

- Given u : **D→C**

$$\begin{tikzcd}
\mathcal{R}\mathbf{C} \arrow[rr, "u^*"] & & \mathcal{R}\mathbf{D} \arrow[ll, bend right=50, "\Sigma_u"'] \arrow[ll, bend left=50, "\Pi_u"] \\
\end{tikzcd}$$

$\perp$ $\perp$

$$\begin{tikzcd}
\mathbf{D} \arrow[d, "u"] & D \arrow[rr, "f"] \arrow[dr, dotted, "v"'] & & D'' \\
& & D' \arrow[ur, dotted, "w"'] & \\
\mathbf{C} & uD \arrow[rr, "uf"] \arrow[dr, "a"'] & & uD'' \\
& & C' \arrow[ur, "b"'] &
\end{tikzcd}$$

weakening of Giraud-Conduché FLP that characterises exponentiable objects of **Cat**/B

Facts:

left adjoint always exists

right adjoint exists iff u satsifies WFLP

LEFT ADJOINTS TO CHANGE OF BASE IN $\mathcal{R}^*(\mathbf{C})$.

$$\begin{tikzcd} \mathcal{R}^*\mathbf{C} \arrow[rr, bend right, "u^*"'] \arrow[rr, phantom, "\perp"] & & \mathcal{R}^*\mathbf{D} \arrow[ll, bend right, "\Sigma_u"'] \end{tikzcd}$$

- Left 2-adjoints always exist
    - Niefield's construction also satisfies universal property wrt the larger class of morphisms

LEFT ADJOINTS TO CHANGE OF BASE IN $\mathcal{R}^*(\mathbf{C})$.

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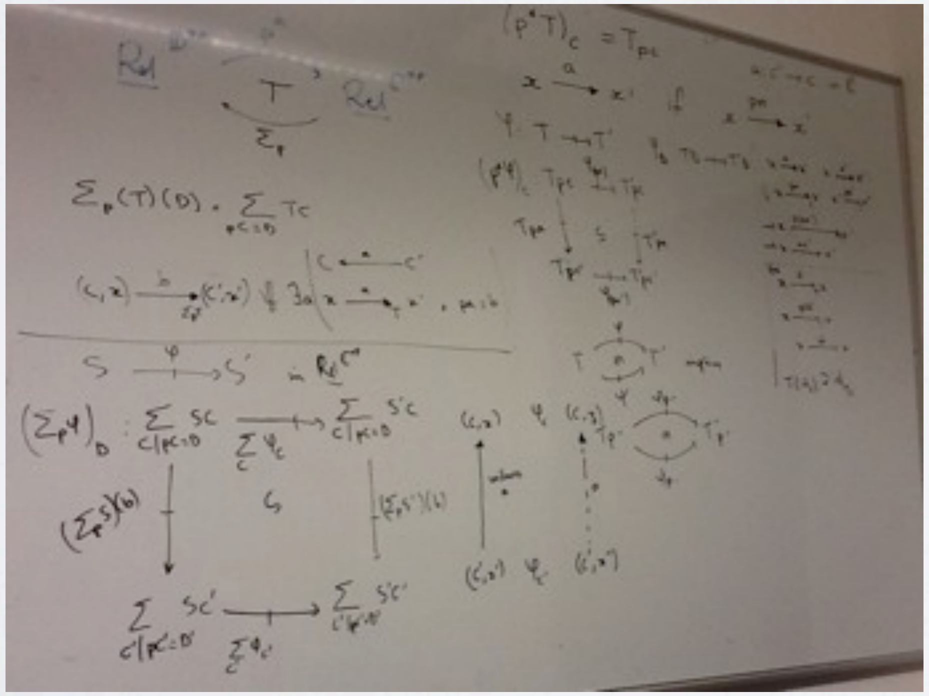
- Left 2-adjoints always exist
  - Niefield's construction also satisfies universal property wrt the larger class of morphisms

Proof:



with help from
Tom Hirschowitz

EXAMPLE

- Let A be a set of actions

$$u\colon A + \{\tau\} \to A^*$$
$$a \longmapsto a$$
$$\tau \longmapsto \epsilon$$

Induces a change of base functor

$$\mathcal{R}^* A^* \xrightarrow{u^*} \mathcal{R}^*(A + \{\tau\})^*$$

intuition: add tau transition for each epsilon transition

LEFT ADJOINT

- 2-adjunction

$$\begin{tikzcd} \mathcal{R}^*(A+\{\tau\})^* \arrow[rr, bend left, "\Sigma_u"] \arrow[rr, phantom, "\perp"] & & \mathcal{R}^* A^* \arrow[ll, bend left, "u^*"] \end{tikzcd}$$

In terms of LTSs, $\boldsymbol{\Sigma}_u$ works by closing wrt:

$$\frac{}{P \xrightarrow{\tau} P} \qquad \frac{P \xrightarrow{\tau} Q \quad Q \xrightarrow{a} R \quad R \xrightarrow{\tau} S}{P \xrightarrow{a} S}$$

and then renaming tau to epsilon

LEFT ADJOINT

- 2-adjunction

$$\begin{tikzcd} \mathcal{R}^*(A+\{\tau\})^* \arrow[r, bend left, "\Sigma_u"] \arrow[r, phantom, "\perp"] & \mathcal{R}^*A^* \arrow[l, bend left, "u^*"] \end{tikzcd}$$

In terms of LTSs, **Σ**$_u$ works by closing wrt:

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and then renaming tau to epsilon

Q. what does the right adjoint look like? (it exists if we restrict to functional morphisms)

WEAK (BI)SIMULATIONS

$$\mathcal{R}^*(A + \{\tau\}) \xrightarrow{\Sigma_u} \mathcal{R}^* A^*$$

- to give a weak simulation from h to h' is to give an arrow

$$\varphi\colon \Sigma_u h \to \Sigma_u h'$$

CONCLUSIONS AND FUTURE WORK

- What I tried to say
  - there are several interesting examples of LTSs that don't seem at home as coalgebras
  - relational presheaves are a natural mathematical universe for such examples
  - some common constructions can be characterised by universal properties