A coalgebraic take on regular and $\omega$-regular behaviour for systems with internal moves

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One of the most fundamental state-based structures considered in computer science literature is a non-deterministic automaton. The class of finite languages accepted by this type of machine with a finite state-space is known under the name of regular languages. On the other hand, these systems have a natural infinite semantics which is given in terms of infinite input satisfying the so-called Büchi acceptance condition (or BAC in short). The condition takes into account the terminal states of the automaton and requires them to be visited infinitely often. It is a common practise to use the term Büchi automaton in order to refer to an automaton whenever its infinite semantics is taken into consideration. The class of infinite languages accepted by finite non-deterministic Büchi automata can be characterized by Kleene theorem for $\omega$-regular languages. Roughly speaking, any such language can be represented in terms of regular languages and the infinite iteration operator $(-)\omega$. Although, the standard type of input of a Büchi automaton is the set of infinite words over a given alphabet, other types (e.g. trees) are also commonly studied and suitable variants of the Kleene theorem hold. This begs the question of a unifying framework these systems can be put in and reasoned about on a more abstract level so that the analogues of Kleene theorems for ($\omega$-)regular input are derived.

The aim of the talk is to present a framework as above for coalgebras with internal moves\(^1\). The approach from [2] suggests that systems with silent steps should be defined as coalgebras whose type is a monad. Although originally, systems with internal moves were modelled as coalgebras $X \rightarrow T(FX + X)$ for a monad $T$ and an endofunctor $F$, such systems can be embedded into coalgebras $X \rightarrow TF^*X$, where $F^*$ is the free monad over $F$ and $TF^*$ itself carries a monadic structure [2]. Unfortunately, the monad $TF^*$ was only tailored to model finite behaviour and is insufficient to cover infinite behaviour. Hence, in the first part of the talk we focus on a description of a monad suitable for our purposes.

The construction of a suitable monad is based on an observation that if an endofunctor $F : C \rightarrow C$ lifts to the Kleisli category for a monad $T$ then the monad $T$ lifts to a monad $\tilde{T} : \text{Alg}(F) \rightarrow \text{Alg}(F)$ on the category of $F$-algebras. The same is true if we replace $\text{Alg}(F)$ with the category $\text{Alg}_B(F)$ of Bloom algebras [1]. The free objects in $\text{Alg}_B(F)$ often exist and are combinations of the free $F$-algebras and the final $F$-coalgebra [1]. Hence, the monad $TF^\infty$ that is suitable to model (in)finite behaviour of systems is defined by composing the

\(^1\) A coalgebraic framework for Büchi automata has been recently developed [3], but it does not take invisible moves into the account and does not reason about ($\omega$-)regular input.
adjunctions $\mathcal{C} \xrightarrow{\mu} \text{Alg}_B(F) \xleftarrow{\nu} \text{Kl}(\hat{T}_B)$ For $T = \mathcal{P}$ and $F = \Sigma^* \times \text{Id}$ we obtain $TF^\omega = \mathcal{P}(\Sigma^* \times \text{Id} + \Sigma^\omega)$.

In the second part of the talk we present a categorical framework to reason about infinite behaviour with BAC. Any non-deterministic (Büchi) automaton without an initial state can be modelled as a pair $(\alpha : X \to \mathcal{P}(\Sigma \times X), \mathfrak{F} \subseteq X)$, where $\mathfrak{F}$ is the set of terminal states. We can extend the codomain of $\alpha$ and consider this map as $F$ where $\mathfrak{F} = \{\varepsilon\}$. Similarily, $\mathfrak{F}$ is uniquely determined by $f_\mathfrak{F} : X \to \mathcal{P}(\Sigma^* \times X + \Sigma^\omega); x \mapsto \text{ if } x \in \mathfrak{F} \text{ then } \{(\varepsilon, x)\} \text{ else } \emptyset$. This means that any such automaton is determined by the pair $(\alpha, f_\mathfrak{F})$ of endomorphisms in the Kleisli category for our monad. Similarly, $\mathfrak{F}$ is uniquely determined by $f_\mathfrak{F} : X \to \mathcal{P}(\Sigma^* \times X + \Sigma^\omega); x \mapsto \text{ if } x \in \mathfrak{F} \text{ then } \{(\varepsilon, x)\} \text{ else } \emptyset$. This means that any such automaton is determined by the pair $(\alpha, f_\mathfrak{F})$ of endomorphisms in the Kleisli category for the above monad. This category is ordered by a complete order, is left distributive and, hence, the following fixpoint operators are well defined: $\alpha^* = \mu_x.\text{id} \lor x \cdot \alpha : X \to X$ and $\beta^\omega = \nu_x.x \cdot \beta : X \to 0$ for $\beta : X \to X$. It can be proven that the map $||(\alpha, \mathfrak{F})|| : X \to \mathcal{P}(\Sigma^\omega)$ (i.e. $||(\alpha, \mathfrak{F})|| : X \to 0$ in the Kleisli category) which assigns to any state $x \in X$ the infinite language with BAC it accepts is given by $||(\alpha, \mathfrak{F})|| = (f_\mathfrak{F} \circ \alpha)^\omega$. This suggests a general definition of the infinite behaviour with BAC for an arbitrary pair of endomorphisms $(\alpha, f)$ in the Kleisli category for $TF^\omega$, where $(-)^\omega$ and $(-)^*$ are well defined.

Now, for any natural numbers $m, n = 0, 1, \ldots$ put $[m] = \{1, \ldots, m\}$ and define $\mathfrak{Reg}(m, n)$ to be the set of morphisms of the form $j \cdot f_\mathfrak{F} \cdot \alpha^* \cdot i$, where $i : [m] \to [k]$ and $j : [k] \to [n]$ are $\text{Set}$-maps, $\mathfrak{F} \subseteq [k]$ and $\alpha : [k] \to T(F[k] + [k])$. Finally, put $\omega \mathfrak{Reg}$ to include maps $(f_\mathfrak{F} \cdot \alpha^*)^\omega \cdot i$ for a $\text{Set}$ map $i : [1] \to [k]$ and $\alpha : [k] \to T(F[k] + [k])$ with $\mathfrak{F} \subseteq [k]$. It can be shown that for $T = \mathcal{P}$ and $F = \Sigma \times \text{Id}$, the set $\mathfrak{Reg}(1, 1)$ and $\omega \mathfrak{Reg}$ are exactly the sets of regular and $\omega$-regular languages respectively in the classical sense. Although it was sufficient for non-deterministic Büchi automata to express behaviours from $\omega \mathfrak{Reg}$ in terms of $\mathfrak{Reg}(1, 1)$ and $(-)^\omega$ it is not enough in general. Hence, we have to consider behaviours from $\mathfrak{Reg}(m, n)$ in order to state the coalgebraic Kleene theorem for $\omega$-regular input. We discuss the conditions under which the following holds.

**Theorem 1 (Kleene theorem for $\omega$-regular behaviour).** $\mathfrak{Reg}$ forms an ordered Lawvere theory which is closed under finite joins, $(-)^\omega$ and it is the smallest subtheory of the Lawvere theory associated with the monad $TF^\omega$ that contains all $\alpha : [k] \to T(F[k] + [k])$ and is closed under finite joins and $(\cdot)^\omega$. Moreover, $\omega \mathfrak{Reg} = \{r_1, \ldots, r_n\} \cdot r + r, r_i \in \mathfrak{Reg}(1, n)\}$. 

**References**


\[\text{Here, } \hat{T}_B \text{ denotes a lifting of } T \text{ to the category of } \text{Alg}_B(F) \text{ Bloom } F\text{-algebras}\]