Alternating Automata via Weak Distributive Laws

Alexandre Goy

Université Paris-Saclay, CentraleSupélec, MICS, France
alexandre.goy@centralesupelec.fr

Abstract

We make use of a weak notion of distributive law to take a second look at the coalgebraic modelling of alternating automata, especially determinization.

In a recent paper from Garner [4] interesting insights are brought in Beck's theory of distributive laws [1]. In order to see the Vietoris monad as a lifting of the powerset monad, Garner is led to make use of a notion of *weak* distributive law already stated in [3].

Definition 1 (Weak distributive law)**.** A *weak distributive law* of a monad $\mathbb{S} = (S, \nu, \omega)$ over a monad $\mathbb{T} = (T, \eta, \mu)$ is a natural transformation $\delta : TS \Rightarrow ST$ such that $\delta \circ \mu S = S\mu \circ \delta T \circ T\delta$, $\delta \circ T\omega = \omega T \circ S\delta \circ \delta S$ and $\delta \circ T\nu = \nu T$.

The usual fourth axiom $\delta \circ \eta S = S\eta$ is simply dropped, for in many cases, this precise axiom is the one inhibiting δ to be a proper distributive law. For example there is no distributive law of the powerset monad $\mathbb{P}$ over the finite distribution monad $\mathbb{D}$ [7], but there is [5] a weak distributive law $\delta : DP \Rightarrow PD$ defined by

$$\delta_X(\Phi) = \left\{\varphi \in DX \mid \exists \Theta \in D(\ni), \forall A \in PX, \Phi(A) = \sum_{x \in A} \Theta(A, x) \text{ and } \forall x \in X, \varphi(x) = \sum_{A \ni x} \Theta(A, x)\right\} \tag{1}$$

Together with this weak variant of distributive laws come weak notions of liftings and extensions of monads [4], along with a bijective correspondence between weak distributive laws, weak extensions, and (whenever idempotents split in the base category) weak liftings.

Definition 2 (Weak lifting)**.** A *weak lifting* of $\mathbb{S}$ to $\mathsf{EM}(\mathbb{T})$ is a monad $\widetilde{\mathbb{S}}$ on $\mathsf{EM}(\mathbb{T})$ along with two natural transformations $\pi : SU^{\mathbb{T}} \Rightarrow U^{\mathbb{T}}\widetilde{S}$, $\iota : U^{\mathbb{T}}\widetilde{S} \Rightarrow SU^{\mathbb{T}}$ such that $\pi \circ \iota = 1$ and the following diagrams commute.

$$\begin{tikzcd}
U^{\mathbb{T}}\widetilde{S}\widetilde{S} \arrow[r, "\iota\widetilde{S}"] \arrow[d, "U^{\mathbb{T}}\widetilde{\mu}^{\mathbb{S}}"'] & SU^{\mathbb{T}}\widetilde{S} \arrow[r, "S\iota"] & SSU^{\mathbb{T}} \arrow[d, "\mu^{\mathbb{S}}U^{\mathbb{T}}"] \\
U^{\mathbb{T}}\widetilde{S} \arrow[rr, "\iota"] & & SU^{\mathbb{T}} \\
& U^{\mathbb{T}} \arrow[ul, "U^{\mathbb{T}}\widetilde{\eta}^{\mathbb{S}}"] \arrow[ur, "\eta^{\mathbb{S}}U^{\mathbb{T}}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
SSU^{\mathbb{T}} \arrow[r, "S\pi"] \arrow[d, "\mu^{\mathbb{S}}U^{\mathbb{T}}"'] & SU^{\mathbb{T}}\widetilde{S} \arrow[r, "\pi\widetilde{S}"] & U^{\mathbb{T}}\widetilde{S}\widetilde{S} \arrow[d, "U^{\mathbb{T}}\widetilde{\mu}^{\mathbb{S}}"] \\
SU^{\mathbb{T}} \arrow[rr, "\pi"] & & U^{\mathbb{T}}\widetilde{S} \\
& U^{\mathbb{T}} \arrow[ul, "\eta^{\mathbb{S}}U^{\mathbb{T}}"] \arrow[ur, "U^{\mathbb{T}}\widetilde{\eta}^{\mathbb{S}}"'] &
\end{tikzcd} \tag{2}$$

Such a weak lifting yields a monad $\widetilde{\mathbb{S}\mathbb{T}} = U^{\mathbb{T}}\widetilde{\mathbb{S}}F^{\mathbb{T}}$ on the base category. Examples include identifying the Vietoris monad [4] (resp. the convex powerset monad [5]) as the *weak* lifting of the powerset monad with respect to the ultrafilter monad (resp. the finite distribution monad).

Alternating automata can be defined as coalgebras for the Set-endofunctor $2 \times (PP-)^A$, where A is an alphabet. The semantics of alternating automata is guided by the following interpretation: an element $\mathcal{A} \in PPX$ is seen as a disjunctive normal form $\bigvee_{U \in \mathcal{A}} \bigwedge_{y \in U} y$, where there is an arbitrary number of clauses of arbitrary length. In concrete terms, given an alternating automaton $\langle o, N\rangle : X \to 2 \times (PPX)^A$:

$$[\![x]\!](\varepsilon) = o(x) \qquad\qquad [\![x]\!](aw) = \bigvee_{U \in N(x)(a)} \bigwedge_{y \in U} [\![y]\!](w) \tag{3}$$

This modelling opens the chase of a possible distributive law of $\mathbb{P}$ over itself. As proved recently [6], this does *not* exist, and actually there is no possible monad structure on PP at all; however it still is possible to model alternating automata coalgebraically, e.g. by looking into Poset [2]. Coming back to our *weak* framework, Garner points out that there is a weak distributive law δ of the finite powerset monad $\mathbb{P}_f$ over $\mathbb{P}$ given by

$$\delta_X(\mathcal{A}) = \{B \subseteq X \text{ finite } \mid B \subseteq \bigcup \mathcal{A} \text{ and } \forall A \in \mathcal{A}, A \cap B \neq \varnothing\} \tag{4}$$

This paves the way for a new modelling of alternating automata as coalgebras for the functor $2 \times (PP_f -)^A$ — modelling that coincides with the usual one for finite systems. Let $G = 2 \times (-)^A$. This modelling yields determinization for alternating automata as in the following diagram:

$$\begin{tikzcd}
\mathsf{Coalg}(GPP_f) \arrow[r, "\widehat{F^{\mathbb{P}_f}}"] \arrow[d] & \mathsf{Coalg}(\widehat{G}\widetilde{P}) \arrow[r, "\widehat{U^{\mathbb{P}_f}}"] \arrow[d] & \mathsf{Coalg}(GP) \arrow[d] \\
\mathsf{Set} \arrow[r, "F^{\mathbb{P}_f}"'] & \mathsf{EM}(\mathbb{P}_f) \arrow[r, "U^{\mathbb{P}_f}"'] & \mathsf{Set}
\end{tikzcd} \tag{5}$$

where $\widehat{G}$ is the lifting of G to $\mathsf{EM}(\mathbb{P}_f)$ arising from the known monad-functor distributive law $\lambda_X(S) = \langle \bigwedge_{(b,f)\in S} b, a \mapsto \{f(a) \mid (b,f) \in S\}\rangle$. The lifted $\widehat{F^{\mathbb{P}_f}}$ consists in transforming a coalgebra $c : X \to GPP_f X$ into $X \xrightarrow{c} GPP_f X \xrightarrow{G\pi_{F^{\mathbb{P}_f}X}} GU^{\mathbb{P}_f}\widetilde{P}F^{\mathbb{P}_f} = U^{\mathbb{P}_f}\widehat{G}\widetilde{P}F^{\mathbb{P}_f}X$ and then taking the adjoint transpose $c^\# : F^{\mathbb{P}_f}X \to \widehat{G}\widetilde{P}F^{\mathbb{P}_f}X$. The lifted $\widehat{U^{\mathbb{P}_f}}$ maps $c : (X,x) \to \widehat{G}\widetilde{P}(X,x)$ to $U^{\mathbb{P}_f}(X,x) \xrightarrow{U^{\mathbb{P}_f}c} U^{\mathbb{P}_f}\widehat{G}\widetilde{P}(X,x) = GU^{\mathbb{P}_f}\widetilde{S}(X,x) \xrightarrow{G\iota_{(X,x)}} GPU^{\mathbb{P}_f}(X,x)$.

Further research includes considering semantics arising from such determinizations, looking at potential weak distributive laws from $\mathbb{P}$ over itself, and investigating compositionality of weak distributive laws in order to model more complex systems.

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