Preservation of Algebraic Features by Monoidal Monads*

Louis Parlant

University College London

Monads are fundamental structures in programming language semantics as they encapsulate many common side-effects such as non-determinism, exceptions, or randomisation. Their structure has been studied not only from an operational point of view but also from a categorical and algebraic perspective. One question that has attracted much attention is that of monad composition: given two monads T and S, is the composition TS again a monad? The answer to the question in full generality is subtle and examples have shown that even in simple cases the correct answer might be surprisingly tricky to find and prove. This subtlety is illustrated, for instance, by Klin and Salamanca's result that the powerset monad does not compose with itself [3], invalidating repeated claims to the contrary in the literature.

Monad composition can be viewed in different ways. On the one hand, a sufficient condition is given by the existence of the categorical notion of a distributive law between the monads or, equivalently, the lifting of the outer monad to the category of Eilenberg–Moore algebras of the inner monad in the composition. On the other hand, if one takes into account the algebraic structure of the inner monad, which can be presented in a traditional operations plus equations fashion, one can turn the original question into a preservation question: does the outer monad preserve all operations and equations of the inner algebra? More generally, starting from a set A with some additional algebraic structure, the question arises naturally: is this structure preserved by the application of a monad? For example, consider a set A that has the structure of a group under with binary operation $\times$ and unit 1. As we apply the *finite powerset monad* $\mathscr{P}$ to form $\mathscr{P}A$, the set of all the finite subsets of A, is $\mathscr{P}A$ is again a group? Can $\times$ be interpreted as a binary operation on elements of $\mathscr{P}A$? Which subset do we identify with the constant 1? In a nutshell, does our monad preserves algebraic features, i.e., the operations and equations defining the structure of A?

This question has already been studied in the late 50s: in [2], Gautam introduces the notion of *complex algebra*—the transformation of any algebra $\mathscr{A}$ with carrier A into an algebra whose elements are the subsets of A. Despite not being described in categorical terms, his construction corresponds exactly to applying a lifting of the monad $\mathscr{P}$ to $\mathscr{A}$. Gautam shows how algebraic operations can be interpreted on $\mathscr{P}A$ and gives a range of positive and negative results for equation preservation. In particular, he shows that commutativity of a binary operation ($x \times y = y \times x$) and unitality ($x \times 1 = x$) are unconditionally preserved by $\mathscr{P}$. These two laws are examples of *linear* equations, in which each variable

* This abstract is based on our arXiv preprint [5]

appears exactly once on each side. Negative results are given for non-linear equations. First, if variables appear more than once (in the sequel, we call these *dup* equations; e.g., idempotency $x \times x = x$), this is not preserved by $\mathscr{P}$. Second, if a variable appears on one side of the equation only (we shall call these *drop* equations, for instance $x \times 0 = 0$), then the powerset does not preserve it either.

In our recent work [5], we examine the more general version of this question: instead of $\mathscr{P}$, we apply an arbitrary monad T to some algebra $\mathscr{A}$. We take as starting point monoidal monads, as these give a canonical way to interpret operations of the algebra on the monad. We then provide a comprehensive characterisation of monad classes and equations that are preserved inside a certain class.

Part of this question has been studied in the literature: in [4], Manes and Mulry show that for a monad to preserve linear equations, it suffices to have a *symmetric monoidal* structure. This argument is further developed in our previous paper [1], where we give sufficient conditions on T for preserving the types of equations outlined by Gautam. In particular, it was shown that so-called *relevant* monads preserve *dup* equations, and *affine* monads preserve *drop* equations. It remained open whether relevance or affineness were necessary conditions for preservation, but the study we undertake in [5] settles the question and establishes the following contributions:

1. We prove that a monoidal monad preserves *drop* equations if and only if it is affine.
2. We characterise a large class of *dup* equations, for which preservation is equivalent to the monad being relevant. We then prove that for a restricted class of *dup* equations, preservation requires a weaker version of relevance, which we call n-relevance.
3. Orthogonally to the semantic characterisations, we prove a more algorithmic result: given a monad and an equation, we show that the general problem of preservation is undecidable.

Our conclusions generalise and extend existing results from [2] and [1]. Furthermore they provide an extensive characterisation of equations preserved by various classes of monoidal monads.

References

1. Fredrik Dahlqvist, Alexandra Silva, and Louis Parlant. Layer by layer: composing monads. In *ICTAC*, 2018.
2. ND Gautam. The validity of equations of complex algebras. *Archiv für Mathematische Logik und Grundlagenforschung*, 3(3-4):117–124, 1957.
3. Bartek Klin and Julian Salamanca. Iterated covariant powerset is not a monad. *MFPS XXXIV*, 2018.
4. Ernie Manes and Philip Mulry. Monad compositions i: general constructions and recursive distributive laws. *Theory and Applications of Categories*, 18(7):172–208, 2007.
5. Louis Parlant, Jurriaan Rot, Alexandra Silva, and Bas Westerbaan. Preservation of equations by monoidal monads. *arXiv preprint arXiv:2001.06348*, 2011.