No-go Theorems for Mixed Distributive Laws

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Monads and Comonads

A **monad** is a triple (M, η, μ) where:

1. $M\colon C \to C$ is an endofunctor.
2. The **unit** $\eta\colon id_C \to M$ is a natural transformation.
3. The **multiplication** $\mu\colon M^2 \to M$ is a natural transformation.

And the following equations hold:

$$\mu \circ M\mu = \mu \circ \mu M; \quad \mu \circ M\eta = \mu \circ \eta M = id_M$$

A **comonad** is a triple (W, ϵ, δ) where:

1. $W\colon C \to C$ is an endofunctor.
2. The **counit** $\eta\colon W \to id_C$ is a natural transformation.
3. The **compultiplication** $\delta\colon M \to M^2$ is a natural transformation.

And the following equations hold:

$$W\delta \circ \delta = \delta W \circ \delta; \quad W\epsilon \circ \delta = \epsilon W \circ \delta = id_W$$

Monads and Comonads

Monads model effectful computation e.g. nondeterminism, probabilities.

An effectful computation from A to B is represented as a morphism $A \to MB$ and can be composed in the **Kleisli** category $Kl(M)$ of a monad M:

- $Obj(Kl(M)) = Obj(C)$
- $Hom_{Kl(M)}(A, B) = Hom(A, MB)$
- $id_x = \eta_x$
- $g \circ f = X \xrightarrow{f} MY \xrightarrow{g^*} MZ$ where:
  - $f : X \to MY$ and $g : Y \to MZ$
  - $g^* = \mu_z \circ Mg$

Comonads model contextual computation e.g. list prefixes, tree nodes.

A contextual computation from A to B is represented as a morphisms $WA \to B$ and can be composed in the **coKleisli** category $coKl(W)$ of a comonad W:

- $Obj(coKl(W)) = Obj(C)$
- $Hom_{coKl(W)}(A, B) = Hom(WA, B)$
- $id_x = \epsilon_x$
- $g \circ f = WX \xrightarrow{f^*} WY \xrightarrow{g} Z$ where:
  - $f : WX \to Y$ and $g : WY \to Z$
  - $f^* = Wf \circ \delta_x$

Natural question: When can computations that are both contextual, and effectful be modelled as morphisms in a suitable category?

Distributive Laws

A (mixed) **distributive law** of a comonad (W, ϵ, δ) over a monad (M, η, μ) is a natural transformation $\lambda : W \circ M \Rightarrow M \circ W$ satisfying four axioms:

Unit

$$\begin{tikzcd} & W \arrow[dl, "W\eta"'] \arrow[dr, "\eta W"] & \\ WM \arrow[rr, "\lambda"'] & & MW \end{tikzcd}$$

Multiplication

$$\begin{tikzcd} WMM \arrow[r, "\lambda M"] \arrow[d, "W\mu"'] & MWM \arrow[r, "M\lambda"] & MMW \arrow[d, "\mu W"] \\ WT \arrow[rr, "\lambda"'] & & MW \end{tikzcd}$$

$$\begin{tikzcd} & WM \arrow[dl, "\lambda"'] \arrow[dr, "\epsilon M"] & \\ MW \arrow[rr, "M\epsilon"'] & & M \end{tikzcd}$$

Counit

$$\begin{tikzcd} WM \arrow[rr, "\lambda"'] \arrow[d, "\delta M"] & & MW \\ WWM \arrow[r, "W\lambda"] & WMW \arrow[r, "\lambda W"] & MWW \arrow[u, "M\delta"'] \end{tikzcd}$$

Comultiplication

Note that each axiom can be satisfied independently of the others. In particular, we say that there exists a **Kleisli law** between W and M whenever the unit and multiplication axioms are satisfied.

BiKleisli Categories

Natural question: When can computations that are both contextual, and effectful be modelled as morphisms in a suitable category?

A mixed distributive law $\lambda : W \circ M \Rightarrow M \circ W$ allows us to define a **biKleisli category** $biKl(W, M)$ whose morphisms are of the form $WA \to MB$:

- $Obj(biKl(W, M)) = Obj(C)$
- $Hom_{biKl(W,M)}(A, B) = Hom(WA, MB)$
- $id_x = \eta_x \circ \epsilon_x$
- $g \circ f = WX \xrightarrow{f^*} WMY \xrightarrow{\lambda_Y} MWY \xrightarrow{g^*} MZ$ where:

$biKl(W, M)$ can be seen as the Kleisli category of M lifted to $coKl(W)$, or equivalently as the coKleisli category of W lifted to $Kl(M)$.

No-Go Theorems

- Distributive laws are not guaranteed to exist, and even when they do, finding them is often difficult.

- [1] Presents several families of no-go-theorems for when the existence of distributive laws between pairs of monads is impossible.

Our contribution: First examples of no-go results for comonad-monad distributive laws.

Inspired by an open question in the game comonads literature:
Do the game comonads G_k distribute over the quantum monad Q_d?

[1] Zwart, Maaike, and Dan Marsden. "No-go theorems for distributive laws." In *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pp. 1-13. IEEE, 2019.

Prefix-List and Power Set

The non-empty powerset monad (P, η, μ) on $\mathbf{SET}$ is given by:

1. $P(X)$ is the set of all non-empty subsets of X.
2. $\eta_X(x)$ is the singleton set$\{x\}$.
3. μ_X takes a union of sets.

The non-empty list comonad (N, ϵ, δ) on $\mathbf{SET}$ is given by:

1. $N(X)$ is the set of all non-empty lists over X.
2. $\epsilon_X[x_1, \ldots, x_n] = x_n$.
3. $\delta_X[x_1, \ldots, x_n] = [[x_1], [x_1, x_2], \ldots, [x_1, x_2, \ldots x_n]]$.

First result: There is no distributive law of the comonad (N, ϵ, δ) over the monad (P, η, μ)

Proof Sketch 1

Some facts:

- [1]: There is a bijective correspondence between:
  1. Kleisli laws $\lambda : W \circ M \Rightarrow M \circ W$
  2. Extensions $\bar{W} : Kl(M) \to Kl(M)$ of W to $Kl(M)$
- $Kl(P)$ is isomorphic to $\mathbf{REL}$.
- [1]: Any weak pullback preserving functor $T : \mathbf{SET} \to \mathbf{SET}$ extends uniquely to a functor $\bar{T} : \mathbf{REL} \to \mathbf{REL}$.
- N preserves weak pullbacks.

[1] Rutten, Jan JMM. "Relators and metric bisimulations." *Electronic Notes in Theoretical Computer Science* 11 (1998): 252-258.

Proof Sketch 2

Corollary: The unique Kleisli law $\lambda : NP \to PN$ has components given by:

$$\lambda_X[X_1, \ldots, X_N] = \{[x_1, \ldots, x_n] \mid x_i \in X_i\}$$

$$\begin{tikzcd}
{[\{a,b\},\{c\}]} \arrow[r, "\lambda_X"] \arrow[d, "\delta_{PX}"'] & \{[a,c],[b,c]\} \arrow[r, "P\delta_X"] & \{[[a],[a,c]],[[b],[b,c]]\} \\
{[[\{a,b\}],[\{a,b\},\{c\}]]} \arrow[r, "N\lambda_X"'] & {[\{[a],[b]\},\{[a,c],[b,c]\}} \arrow[r, "\lambda_{NX}"'] & \{[[a],[a,c]],[[b],[a,c]],[[a],[b,c]],[[b],[b,c]]\},
\end{tikzcd}$$

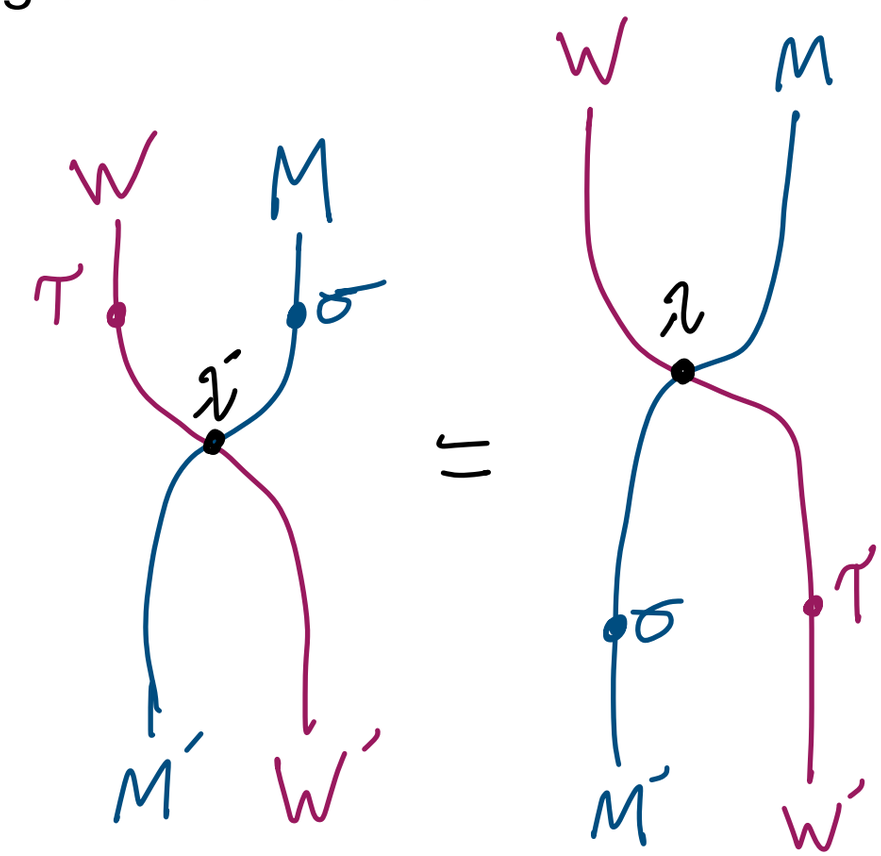
$$\neq$$

Transfer Theorems

Theorem : Assume the following diagram commutes:

$$\begin{tikzcd} WM \arrow[r, "\lambda"] \arrow[d, "\tau\sigma"'] & MW \arrow[d, "\sigma\tau"] \\ W'M' \arrow[r, "\lambda'"] & M'W' \end{tikzcd}$$

or

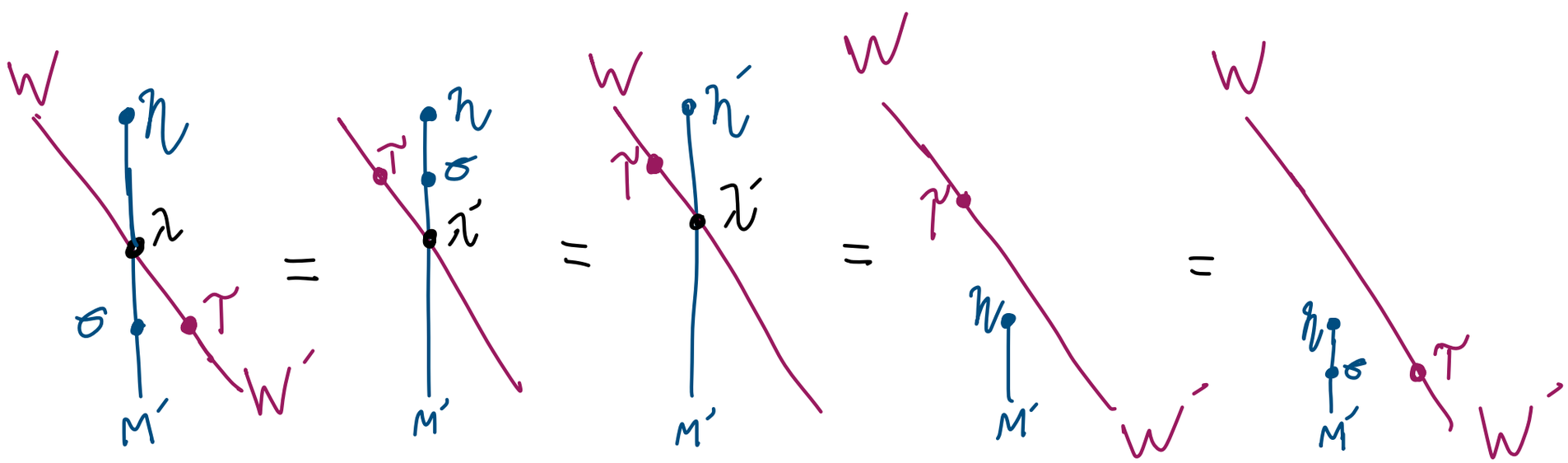


Then we have (with some caveats):

1. If λ' is a distributive law and τ, σ are monic, then λ is a distributive law.
2. If λ is a distributive law and τ, σ are epic, then λ' is a distributive law.

Proof sketch

- Transfer theorems can be proven elegantly using string diagrams:



Example: Stream Comonad

- The stream with suffixes comonad (S, ϵ, δ) on $\mathbf{SET}$ is given by:
  1. $S(X)$ is the set of all infinite sequences over X.
  2. $\epsilon_X(x_1, x_2, \dots) = x_1$.
  3. $\delta_X(x_1, x_2, \dots) = ((x_1, x_2, \dots), (x_2, x_3, \dots), (x_3, x_4, \dots), \dots)$.

- S preserves weak pullbacks. Thus there exists a unique kleisli law $\lambda' : SP \to PS$.
- There exists a monic comonad morphism $\tau : N \to S$:

$$\tau[x_1, x_2, \dots, x_n] = (x_n, x_{n-1}, \dots x_1, x_1, x_1, \dots)$$

- Moreover, the identity monad map $id : P \to P$ is clearly monic.
- Hence, the conditions of our transfer theorem are satisfied and we have:

Prop: If $\lambda' : SP \to PS$ is a distributive law then $\lambda : NP \to PN$ is a distributive law.

Corollary: There is no distributive law of the comonad S over the monad P.

Generalisation to $\mathscr{M}_S$

- Let S be a commutative semiring. The multiset monad for S, $(\mathscr{M}_S, \eta^D, \mu^D)$ is a monad on $\mathbf{SET}$ given by:
  1. $\mathscr{M}_S(X) = \{\varphi : X \to S \mid \text{supp}(\varphi) \text{ is finite}\}$
  2. $\eta_X(x) = 1x$
  3. $\mu_X(\sum_i s_i\varphi_i)(x) = \sum_i s_i \,.\, \varphi_i(x)$
- P arises as the special case of $\mathscr{M}_S$ where S is the boolean semiring.

There is no comonad-monad distributive law between N and M_S for any S.

A categorical proof (work in progress)

- What we want:

The unique Kleisli law $\lambda\colon NP \to PN$ has components given by:
$$\lambda_X[X_1, \ldots, X_N] = \{[x_1, \ldots, x_n] \mid x_i \in X_i\}$$

- We define a new category $S - Rel$ whose objects are sets.

- Morphisms of $\mathbf{S - REL}$ are tuples (R, w) such that the following is a pullback.

$$\begin{tikzcd} R \arrow[r, hook, "\iota"] \arrow[d, "w|_{U(S)\setminus\{0_S\}}"'] & X \times Y \arrow[d, "w"] \\ U(S) \setminus \{0_S\} \arrow[r, hook] & U(S) \end{tikzcd}$$

- We need to check:

1. $\mathbf{S - REL}$ is isomorphic to $Kl(M_S)$
2. Any comonad uniquely extends to an endofunctor over $\mathbf{S - Rel}$

Other results

We did not have time to cover the following results. They will appear in the full version of our paper.

1. Generalisation to the distribution monad over arbitrary semirings.

2. Generalisation to game comonads in the category of relational structures.

3. Generalisation to larger classes of weak pullback preserving comonads (Directed Containers?)