Coinductive control of inductive data types

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A coalgebra C which associates to every element c a possible collection of elements $(c_{(1)}, c_{(2)})$, and define a function $C \to [A, B]$ such that, for each $c \in C$, the map $f_c\colon A \to B$:

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Idea appears in 1968 by Sweedler in the context of vector spaces.

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Gain

Get more control over algebras

- Get more "initial algebras" (e.g. generalized W-types)

Previous work on coalgebraic enrichment

Univeral measuring coalgebra (Wraith, Sweedler 1968)

For k-algebras A and B, there is a k-coalgebra $\underline{\mathrm{Alg}}(A, B)$

- which underlies an enrichment of k-algebras in k-coalgebras
- whose *set-like elements*[1] are in bijection with $\mathrm{Alg}(A, B)$.

Taking $B := k$, one gets the *dual* $\underline{\mathrm{Alg}}(A, k)$ of A.

Extensions

- Anel-Joyal 2013 (dg-algebras)
- Hyland-Franco-Vasilakopoulou 2017 (monoids)
- Vasilakopoulou 2019 ($\mathcal{V}$-categories)
- P. 2022 (∞-algebras of an ∞-operad)
- McDermott-Rivas-Uustalu 2022 (monads)
- North-P 2023 (algebras of endofunctor)

[1] those $c \in \underline{\mathrm{Alg}}(A, B)$ s.t. $\Delta c = c \otimes c$ and $\epsilon(c) = 1_A$

Review of categorical W-types

Let $\mathcal{C}$ be a locally presentable, symmetric monoidal closed category, i.e. Set with its usual Cartesian monoidal structure.

Natural numbers

The type of natural numbers $\mathbb{N}$ is the initial algebra for the endofunctor $X \mapsto X + 1$.

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Lists

The type of lists $\mathbb{L}\text{ist}(A)$ is the initial algebra for the endofunctor $X \mapsto X \times A + 1$.

When A is equipped with the structure of a commutative monoid, this fulfills our hypotheses.

Enriched categories

Definition

An *enrichment* of a category $\mathcal{C}$ in a monoidal category $\mathcal{V}$ consists of

- a functor $\underline{\mathcal{C}}(-,-) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{V}$
- a morphism $\mathbb{I} \to \underline{\mathcal{C}}(A, A)$ for each object A of $\mathcal{C}$
- a morphism $\underline{\mathcal{C}}(A, B) \otimes \underline{\mathcal{C}}(B, C) \to \underline{\mathcal{C}}(A, C)$ for each triple A, B, C of objects of $\mathcal{C}$
- an isomorphism $\mathcal{V}(\mathbb{I}, \underline{\mathcal{C}}(A, B)) \cong \mathcal{C}(A, B)$

Remark

Monoidal *closed* means enriched in itself.

Outline

Introduction and background

Endofunctors

Measuring in general

Fix a category $\mathcal{C}$ and an endofunctor F satisfying our hypotheses.

Measuring

Given algebras $(A, FA \xrightarrow{\alpha} A), (B, FB \xrightarrow{\beta} B)$ a *measuring* $(A, \alpha) \to (B, \beta)$ is a coalgebra $(C, C \xrightarrow{\chi} FC)$ together with a morphism $\phi : C \to \underline{\mathcal{C}}(A, B)$ such that

$$\begin{array}{ccccccc} & \overset{\chi}{\nearrow} & FC & \xrightarrow{F(\phi)} & F(\underline{\mathcal{C}}(A,B)) & \longrightarrow & \underline{\mathcal{C}}(FA, FB) \\ C & & & & & & \downarrow \beta \\ & \underset{\phi}{\searrow} & \underline{\mathcal{C}}(A,B) & & \xrightarrow{\quad\quad\alpha\quad\quad} & & \underline{\mathcal{C}}(FA, B) \end{array}$$

i.e., the measure and the co/algebra structures are compatible.

The *universal measuring* $\underline{\mathrm{Alg}}(A, B)$ is the terminal measuring $(A, \alpha) \to (B, \beta)$.

Measuring for the natural numbers

Consider the endofunctor $X \mapsto X + 1$ on Set.

- Algebras are sets A together with $A + 1 \to A$
  - Have $-_A : \mathbb{N} \to A$
- Coalgebras are sets C together with $C \to C + 1$
  - Have $[\![-]\!] : C \to \mathbb{N}^\infty$

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Recall a homomorphism $f : A \to B$ between algebras is a function such that $f(0_A) = 0_B$ and $f(a + 1) = f(a) + 1$, for all $a \in A$.

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Measuring

For algebras A, B, a *measuring* $A \to B$ is a coalgebra C together with a function $C \to A \to B$ such that

- $f_c(0_A) = 0_B$ for all $c \in C$;
- $f_c(a + 1) = 0_B$ for all $[\![c]\!] = 0$ and for all $a \in A$;
- $f_c(a + 1) = f_{c-1}(a) + 1$ for $[\![c]\!] \geqslant 1$ and for all $a \in A$.

The *universal measuring* $\underline{\mathrm{Alg}}(A, B)$ is the terminal measuring $A \to B$.

Set-like elements in general

Definition

The *set-like elements* are coalgebra homomorphisms

$$\mathbb{1} \to \underline{\mathrm{Alg}}(A, B)$$

i.e., elements of $\mathrm{Alg}(A, B)$.

Set-like elements for the natural numbers

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Example

$$\mathrm{Alg}(\mathbb{N}, A) \cong *$$
$$\underline{\mathrm{Alg}}(\mathbb{N}, A) \cong \mathbb{N}^{\infty}$$

What are the non-set-like elements?

Example

$$\mathrm{Alg}(\mathbb{N}, A) \cong *$$
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The elements corresponding to $n \in \mathbb{N}^\infty$ are functions which 'are algebra homomorphisms' on $\{0, \dots, n\} \subseteq \mathbb{N}$, i.e., are *n-partial homomorphisms*.

- Let $\mathbb{n}$ denote the quotient of $\mathbb{N}$ by $m = n$ for all $m \geqslant n$.
- Let $\mathbb{n}^\circ$ denote the subobject of $\mathbb{N}^\infty$ consisting of $\{0, \dots, n\}$.

Example

$$\underline{\mathrm{Alg}}(\mathbb{n}, A) \cong \begin{cases} \mathbb{N}^\infty & \text{if } n_A = m_A \text{ for all } m \geqslant n; \\ \mathbb{n}^\circ & \text{otherwise.} \end{cases}$$

What can we do with this?

Perhaps define more general *initial objects*.

C-initial objects

For a coalgebra C, a *C-initial algebra* is an algebra A universal with the property that for all other algebras B there is a unique

$$C \to \underline{\mathsf{Alg}}(A, B).$$

Examples

For the natural-numbers endofunctor:

- $\mathbb{N}$ is the $\mathbb{I}$*-initial algebra*
- $\mathbb{N}$ is the $\mathbb{N}^{\infty}$*-initial algebra*
- $\mathbb{n}$ is the $\mathbb{n}^{\circ}$*-initial algebra*

Future work

- Work out more examples in detail
- Understand what it means to endow the containers with extra stucture (e.g. A needs a commutative monoid structure for the container for $\mathbb{L}\text{ist}(A)$)
- Understand C-initial algebras in more examples and in general
- Understand if this extra structure is useful for programming languages

Thank you!