Structural Operational Semantics for Heterogeneously Typed Coalgebras

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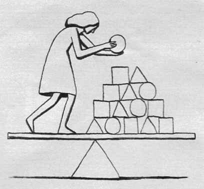


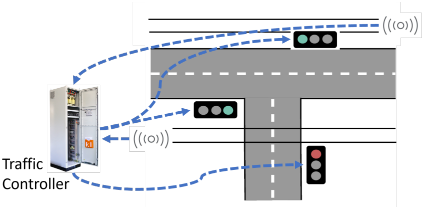
Background and Research Question

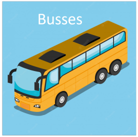
- Joint Research Program at UiB and HVL, Bergen
  - Coordination of concurrently interacting components in modular software architectures
  - Purpose: Correctness checks w.r.t. global consistency rules and ...
  - ... later: (Semi-)automatic repair (proposals)
- State of the project
  - Component alignment generates amalgamated graph transformation system[1]
  - But carried out by ad-hoc implementations depending on the involved behaviours

What is a general coordination framework for capturing the dynamics of interacting, arbitrarily typed behavioural components?

[1] T. Kräuter, H. König, Y. Lamo, A. Rutle, P. Stünkel: *Towards Behavioral Consistency in Multi-Modeling*, to appear in JOT, 2023

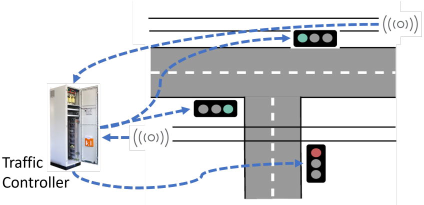
Traffic Control: (Local) Components

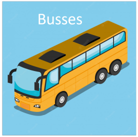




- Traffic Controller: Requests traffic light changes, receives sensor signals
- Traffic Lights: Change state passively
- Buses: Probabilty distribution in simulation scenario
- Tram: Timed Processes

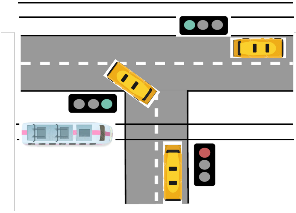
Traffic Control: Compound System



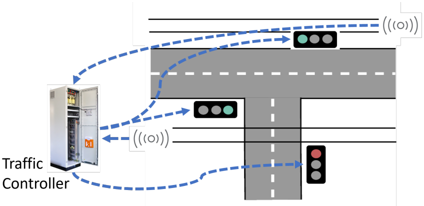


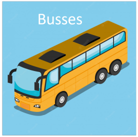
Coordination Language, Generation of Compound Behaviour

$\Downarrow$



Coalgebras for General Dynamical Systems

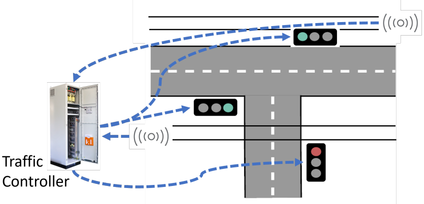


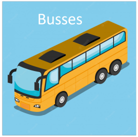


Heterogeneous behaviour: Different $\mathcal{SET}$-endofunctors

- Traffic Controller: $\mathcal{B}_1 = (1 + O \times \_)^E$
- Traffic Lights: $\mathcal{B}_2 = (1 + \_)^I$
- Buses: $\mathcal{B}_3 = (1 + \mathcal{D}_\omega(\_))$
- Tram: $\mathcal{B}_4 = (\_)^\mathcal{T}$
- Compound System: $\mathcal{B} = ?$

Generating Global Behaviour





- Green light, tram approaching ⇒ Red light, tram passing
- I.e. inference rule

$$\frac{\text{premises}}{\text{conclusion}}$$

Capturing Compound Behaviour: SOS-Laws

For $\mathcal{B}$ an intended behavioural interpretation, Σ process term syntax, e.g.

$$\frac{}{a.x \xrightarrow{a} x} \quad \frac{x \xrightarrow{a} x' \qquad y \xrightarrow{b} y'}{x||y \xrightarrow{a} x'||y} \quad \frac{x \xrightarrow{a} x' \qquad y \xrightarrow{b} y'}{x||y \xrightarrow{b} x||y'}, a, b \in A.$$

In the above example:

- $\Sigma(X) = 1 + \coprod_{a \in A} X + X^2$
- $\mathcal{B}(X) = \wp_{fin}(X)^A$, $\mathcal{H}(X) := X \times \mathcal{B}(X)$

Encoding of rules (shown only for $||$):

$$\rho_X : \begin{cases} (X \times \wp_{fin}(X)^A)^2 & \to \wp_{fin}(1 + \coprod_{a \in A} X + X^2)^A \\ (x, \beta_1, y, \beta_2) & \mapsto \lambda c.\{(x', y) \mid x' \in \beta_1(c)\} \cup \{(x, y') \mid y' \in \beta_2(c)\} \end{cases}$$

Similarly for the other summands of Σ: $\rho : \Sigma\mathcal{H} \Rightarrow \mathcal{B}\Sigma$.

$$\boxed{\Sigma\text{-}Terms\ of\ \mathcal{B}ehaviour \Rightarrow \mathcal{B}ehaviour\ of\ \Sigma\text{-}Terms}$$

Capturing Compound Behaviour: Distributive Laws

Definition (Distributive Law)

Let $\mathbb{C}$ be a category with products, $\Sigma, \mathcal{B} : \mathbb{C} \to \mathbb{C}$ and $\mathcal{H}(X) := X \times \mathcal{B}(X)$. A *Distributive Law of* Σ *over* $\mathcal{H}$ is a natural transformation $\lambda : \Sigma\mathcal{H} \Rightarrow \mathcal{H}\Sigma$, which is compatible with $(\pi_{1,X} : X \times \mathcal{B}(X) \to X)_{X \in \mathcal{SET}}$, i.e. $\pi_{1,\Sigma} \circ \lambda = \Sigma\pi_1$.

Theorem

Natural Transformations $\rho : \Sigma\mathcal{H} \Rightarrow \mathcal{B}\Sigma \overset{1:1}{\leftrightarrow}$ *Distributive Laws over* $\mathcal{H}$.

Remark:

Theorem (Turi, Plotkin, 1997)

For the above example, image-finite GSOS rule sets are in 1:1-correspondence to natural transformations $\Sigma\mathcal{H} \Rightarrow \mathcal{B}\Sigma^*$.

Compositionality and Bialgebras

- $(x_i \sim x'_i)_{i \in \{1,\ldots,n\}} \Rightarrow op(x_1, \ldots, x_n) \sim op(x'_1, \ldots, x'_n)$
- In our context: Is observational equivalence preserved after the construction of the compound system?

For $\lambda : \Sigma\mathcal{H} \Rightarrow \mathcal{H}\Sigma : \mathbb{C} \to \mathbb{C}$ and $X \xrightarrow{\alpha} \mathcal{H}(X)$ there is the λ-induced coalgebra

$$\Sigma(X) \xrightarrow{\Sigma\alpha} \Sigma\mathcal{H}(X) \xrightarrow{\lambda_X} \mathcal{H}\Sigma(X)$$

Furthermore, there are two important diagrams:

$$\begin{array}{ccccc} \Sigma(A) & & \xrightarrow{init} & & A \\ \downarrow{\scriptstyle \Sigma h_\lambda} & & & & \downarrow{\scriptstyle h_\lambda} \\ \Sigma\mathcal{H}(A) & \xrightarrow[\lambda_A]{} & \mathcal{H}\Sigma(A) & \xrightarrow[\mathcal{H}\,init]{} & \mathcal{H}(A) \end{array}
\qquad
\begin{array}{ccccc} \Sigma(A) & \xrightarrow{init} & A & \xrightarrow{h_\lambda} & \mathcal{H}(A) \\ \downarrow{\scriptstyle \Sigma f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathcal{H}f} \\ \Sigma(Z) & \xrightarrow{g_\lambda} & Z & \xrightarrow{\zeta} & \mathcal{H}(Z) \end{array}$$

Theorem (Klin, 2011)

Observational Equivalence is a Congruence.

Challenges, Refined Research Question

Facts:

- Inference rules inductively determine provable behaviour of *all* Σ-terms from atomic transitions.
- Bialgebraic Theory provides general proof for *compositionality*.
- Transition rules act on a *single state space*.

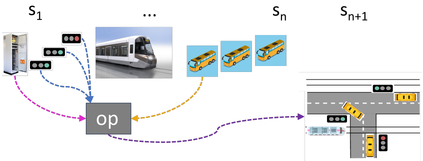
Challenges:

- We want to avoid recurrent term generation and *only* determine behaviour of the generated compound system.
- With an adjusted approach for interacting individual components, can we still guarantee *compositionality*?
- In *heterogeneous* specifications, *state spaces* must be kept *separate*!

How can we apply the bialgebraic theory to formally understand interacting, heterogeneously typed behavioural components?

Many-Sortedness and Holistic Approach

Non-recurrent "term"-generation via different sorts:



- Each "attempt" to let certain components interact, is represented by an operation $op : s_1 \cdots s_n \to s_{n+1}$
- In an $n+1$-sorted algebra, each carrier of sort $s \in \{s_1, .., s_n\}$ represents the state space of a local component, and ...
- ... the carrier for sort $n+1$ represents the state space of the compound system.

→ Algebras simultaneously describe the local and global state spaces.

Interaction Law Instead of Distributive Laws

Let $((S_i, \alpha_i) \in \mathcal{B}_i\text{-}\mathcal{C}oalg)_{i \in \{1,\dots,n\}}$ be the local components and $\mathcal{B}$ be the behavioural specification of the compound system.
Keeping state spaces separate in a rule-induced coalgebra ($n = 2$):

$$X_1 \times X_2 \xrightarrow{\langle id, \alpha_1 \rangle \times \langle id, \alpha_2 \rangle} X_1 \times \mathcal{B}_1(X_1) \times X_2 \times \mathcal{B}_2(X_2) \xrightarrow{\rho_{X_1, X_2}\ ?} \mathcal{B}(X_1 \times X_2)$$

In general $X_1 \times X_2$ is replaced by an arbitrary set ($\mathbb{C}$-object) constructed out of n input sets:

$$\Sigma : \mathcal{SET}^n \to \mathcal{SET}$$

Definition (Interaction Law)

An *interaction law* is a natural transformation

$$\rho : \Sigma(\mathcal{H}_1 \times \cdots \times \mathcal{H}_n) \Rightarrow \mathcal{B}\Sigma : \mathcal{SET}^n \to \mathcal{SET}.$$

But this is apparently no longer an interplay between endofunctorial syntax and a single behaviour!

Example: Heterogeneous SOS

$$\left(\frac{x_1 \xrightarrow{e/o} x_1' \qquad x_2 \xrightarrow{i} x_2'}{op(x_1, x_2) \xrightarrow{\varphi(o,i)} op(x_1', x_2')} \right)_{o \in O, i \in I}$$

As interaction law:

$$\rho_{X_1,X_2} : X_1 \times (1 + O \times X_1)^E \times X_2 \times (1 + X_2)^I \to \wp_{fin}(X_1 \times X_2)^{Act}$$

$$(x_1, \beta_1, x_2, \beta_2) \mapsto \begin{cases} \{(\tau, (x_1', x_2')) \mid & (o, x_1') \in \beta_1(E), x_2' = \beta_2(i)\} \\ \cup & \\ \{(o, (x_1', x_2)) \mid & (o, x_1') \in \beta_1(E), o \text{ unsynchr.}\} \\ \cup & \\ \{(i, (x_1, x_2')) \mid & x_2' = \beta_2(i), i \text{ unsynchr.}\} \end{cases}$$

The Main Result

Uses ▸ classical results for endofunctors despite state space separation!

Theorem

Let $\left(S_i \xrightarrow{\alpha_i} \mathcal{B}_i(S_i) \in \mathcal{B}_i\text{-}\mathcal{C}oalg\right)_{i \in \{1,\ldots,n\}}$ and $\mathcal{B}$ be the behavioural specification of the compound system. Let them all admit final objects. Let $\Sigma : \mathcal{SET}^n \to \mathcal{SET}$. Compositionality holds, if the computation of the compound system can be described by an interaction law $\rho : \Sigma(\mathcal{H}_1 \times \cdots \times \mathcal{H}_n) \Rightarrow \mathcal{B}\Sigma$.

Proof idea:

- Holistic behaviour: $\vec{\mathcal{B}} := \Pi_{1 \leq i \leq n} \mathcal{B}_i \times \mathcal{B} : \mathcal{SET}^{n+1} \to \mathcal{SET}^{n+1} =: \mathbb{C}$
- Lifting local behaviour to global behaviour: $\vec{\Sigma} : \mathbb{C} \to \mathbb{C}$ def. by $\vec{\Sigma}(X_1, \ldots, X_n, X_{n+1}) := (S_1, \ldots, S_n, \Sigma(X_1, \ldots, X_n))$
- Fixing locals: $\vec{\rho} := (\alpha_1, \ldots, \alpha_n, \rho) : \vec{\Sigma}\vec{\mathcal{H}} \Rightarrow \vec{\mathcal{B}}\vec{\Sigma} : \mathbb{C} \to \mathbb{C}$ ▸ Option
- Thus setting of ▸ 10 , which yields distributive law $\vec{\lambda} : \vec{\Sigma}\vec{\mathcal{H}} \Rightarrow \vec{\mathcal{H}}\vec{\Sigma}$ and with ▸ classical results the desired result. □

Related Work

- Practical Approaches
  - Co-simulation
  - Coordination Languages
- Coalgebraic Abstraction of SOS Framework
  - Klin's Survey
  - Categorically in B. Jacobs' book
- Heterogeneity
  - M. Kick, J. Power, A. Simpson *Coalgebraic semantics for timed processes.*
  - ...
- (Co-)Institutions
- See references in the paper

Résumée and Future Work

Holistic many-sorted formal approach for concurrently interacting heterogeneously typed coalgebras. Evaluation of the approach by proving compositionality.

Future Work:

- Implementation viewpoint: Currently very cumbersome
- Intermediate interaction: Sort inflation
- Extensions: $\mathcal{B}$ehaviour: Name passing, Σyntax: Equational specifications
- Adequate (co-)institutional methods
- Aspects of Temporal Constraints

Optional Slide to Explain Action of $\vec{\rho}$

$(S_i \xrightarrow{\alpha_i} \mathcal{B}(S_i) \in \mathcal{B}_i\text{-}\mathcal{C}oalg)_{i\in\{1,\dots,n\}}, \rho : \Sigma(\mathcal{H}_1 \times \cdots \times \mathcal{H}_n) \Rightarrow \mathcal{B}\Sigma.$

- $\vec{\mathcal{B}} := \Pi_{1\leq i\leq n}\mathcal{B}_i \times \mathcal{B} : \mathcal{SET}^{n+1} \to \mathcal{SET}^{n+1}$
- $\vec{\mathcal{H}} := \Pi_{1\leq i\leq n}\mathcal{H}_i \times \mathcal{H} : \mathcal{SET}^{n+1} \to \mathcal{SET}^{n+1}$
- $\vec{\Sigma}(X_1, \dots, X_n, X_{n+1}) := (S_1, \dots, S_n, \Sigma(X_1, \dots, X_n))$

Then it is defineable as

$$\vec{\rho}_{X_1,\dots,X_{n+1}} : \vec{\Sigma}\vec{\mathcal{H}}(X_1, \dots, X_{n+1}) \to \vec{\mathcal{B}}\vec{\Sigma}(X_1, \dots, X_{n+1})$$

because

$$\begin{array}{lccccccc} \vec{\rho}_{X_1,\dots,X_{n+1}} & : & S_1 & \times\cdots\times & S_n & \times & \Sigma(\mathcal{H}_1(X_1), \dots, \mathcal{H}_n(X_n)) \\ & & \downarrow \alpha_1 & \cdots & \downarrow \alpha_n & & \downarrow \rho_{X_1,\dots,X_n} \\ & \to & \mathcal{B}_1(S_1) & \times\cdots\times & \mathcal{B}_n(S_n) & \times & \mathcal{B}\Sigma(X_1, \dots, X_n) \end{array}$$

i.e. $\vec{\rho} := (\alpha_1, \dots, \alpha_n, \rho) : \vec{\Sigma}\vec{\mathcal{H}} \Rightarrow \vec{\mathcal{B}}\vec{\Sigma}.$

Optional: Adapted Notion of Congruence

Let $A_1, \ldots, A_n, A$ be sets and

$$f : \Sigma(A_1, \ldots, A_n) \to A$$

be a map. A family of binary relations

$$(R_i \subseteq A_i \times A_i)_{i \in \{1,\ldots,n\}}, R \subseteq A \times A$$

is said to be f-*compatible*, if there is a map r, such that the following diagram commutes:

$$\begin{array}{ccccc}
\Sigma(A_1, \ldots, A_n) & \xleftarrow{\Sigma(\pi_1^1, \ldots, \pi_1^n)} & \Sigma(R_1, \ldots, R_n) & \xrightarrow{\Sigma(\pi_2^1, \ldots, \pi_2^n)} & \Sigma(A_1, \ldots, A_n) \\
\downarrow f & & \downarrow r & & \downarrow f \\
A & \xleftarrow[\pi_1]{} & R & \xrightarrow[\pi_2]{} & A
\end{array}$$