Semantics of Multimodal Adjoint Type Theory

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Unimodal type theories

A unimodal type theory consists of

1. An ordinary type theory
2. Some new unary type formers, called modalities.
3. Specified functions relating the modalities and their composites, today called laws.

Example (Classical modal logic)

Two modalities:

- $\Box P$ = "necessarily P" (P holds in all possible worlds)
- $\Diamond P$ = "possibly P" (P holds in some possible world)

Laws including

$$\Box P \to P \qquad P \to \Diamond P \qquad \Box P \to \Box\Box P \qquad \text{etc.}$$

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- Phase distinctions: modalities $\circ_\phi$, $\bullet_\phi$, laws $A \to \circ_\phi A$ and $\circ_\phi \circ_\phi A \cong \circ_\phi A$, sim. $A \to \bullet_\phi A$ and $\bullet_\phi \bullet_\phi A \cong \bullet_\phi A$, etc.

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- Synthetic topology: Both $\flat$ and $\sharp$, with $\flat\sharp A \cong \flat A$, $\sharp\flat A \cong \sharp A$. Any type has an implicit topology, e.g. $\mathbb{N} \to \mathbf{2}$ is Cantor space. Then $\flat A$ retopologizes it discretely, and $\sharp A$ indiscretely.

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- Enhanced guarded recursion: modalities $\triangleright$ and $\Box$ ("always"), laws $\Box A \to A$ and $\Box\Box A \cong \Box A$ (like $\flat$) plus $\Box \triangleright A \cong \Box A$.

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- Similarly, all the $\flat$s and $\sharp$s can be decomposed through sets.
- Call by push value*: modes v (values) and c (computations), modalities $\mathrm{F} : v \to c$ and $\mathrm{U} : c \to v$, laws giving $\mathrm{F} \dashv \mathrm{U}$.

General modal type theories

The abstract structure of modes, modalities and laws forms a 2-category, with objects, morphisms, and 2-cells.

Big-picture goal

Formulate and implement a general multimodal type theory, parametrized over a user-specified 2-category $\mathcal{M}$.

A very biased and selective history:

- Pfenning–Davies 2001: unimodal simple type theory with $\Box$
- Reed 2009: multimodal simple type theory over any poset $\mathcal{M}$
- Licata–S.–Riley 2017: multimodal simple type theory over any 2-category $\mathcal{M}$
- S. 2018: unimodal dependent type theory with $\flat, \sharp$
- Gratzer–Kavvos–Nuyts–Birkedal 2021: multimodal dependent type theory (MTT) over any 2-category $\mathcal{M}$

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- $\mu \boxdot A$ internalizes annotated variables, with elimination rule

$$\frac{\Gamma \vdash M : \mu \boxdot A \qquad \Gamma, x :^{\mu} A \vdash c : C}{\Gamma \vdash \mathsf{let\,mod}(x) \leftarrow M \,\mathsf{in}\, c : C}.$$

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- The introduction rule requires a "locking" or "division" operation making a q-context into a p-context:

$$\frac{\color{red}{\Gamma/\mu} \vdash M : A}{\Gamma \vdash \mathsf{mod}(M) : \mu \boxdot A}$$

Context operations

In PD, Reed, LSR, etc., $\Gamma/_{\mu}$ was computed by removing or re-annotating variables according to the laws, e.g.

$$(x :^{\flat} A, y : B, z :^{\flat} C)/_{\flat} = (x :^{\flat} A, z :^{\flat} C)$$

Here $y : B$ (meaning $y :^{\mathrm{id}} B$) is removed as there is no law $\mathrm{id} \Rightarrow \flat$.
In general, $(x :^{\mu} A)/_{\nu}$ contains an $x :^{\varrho} A$ for each $\alpha : \mu \Rightarrow \nu \circ \varrho$.

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In MTT, $\Gamma/_\mu$ is a constructor of contexts: a context is a sequence of variables interspersed with "formal divisions" (a.k.a. "locks"). Now we choose a law when using a variable, e.g.

$$\frac{\alpha : \mu \Rightarrow \nu}{\Gamma, (x :^\mu A),\ /_\nu,\ (y : B) \vdash x^\alpha : A}$$

Semantics of modal type theory

Multimodal type theory over $\mathcal{M}$ should have semantics in a 2-functor $\mathcal{M} \to \mathcal{C}at$:

1. Each mode p represents a (structured) category $\mathscr{C}_p$.
2. Each modality $\mu \boxdot -$ represents a functor $\mathscr{C}_p \to \mathscr{C}_q$.
3. Each law represents a natural transformation.

Example (Guarded recursion)

$\mathscr{C}_t = \mathrm{Set}^{\omega^{\mathrm{op}}}$ (the "topos of trees") and $\mathscr{C}_s = \mathrm{Set}$, with

$$\triangleright X(n) = X(n+1) \qquad \mathtt{tot}\, X = \lim_n X(n) \qquad (\mathtt{const}\, Y)(n) = Y$$

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Some functors do have left adjoints:

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- open modalities $\bigcirc_\phi$
- indiscreteness $\sharp$ in topological spaces
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- open modalities $\circ_\phi$
- indiscreteness $\sharp$ in topological spaces
- discreteness $\flat$ in locally connected topological spaces

But others don't:

- $\sharp$ in the effective topos
- closed modalities $\bullet_\phi$
- tangent space T
- global sections $\flat$ in a general topos
- discreteness $\flat$ in non-locally-connected topological spaces

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Given $\mathscr{C} : \mathcal{M} \to \mathcal{C}at$, let an object of $\widehat{\mathscr{C}}_r$ consist of

1. For each $\mu : p \to r$ in $\mathcal{M}$, an object $\Gamma_\mu \in \mathscr{C}_p$.
2. For each $\varrho : p \to q$ and $\alpha : \mu \Rightarrow \nu \circ \varrho$, a map $\Gamma_\nu \to \mathscr{C}_\varrho(\Gamma_\mu)$.
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Theorem

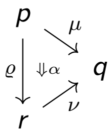
If each $\mathscr{C}_p$ has, and each $\mathscr{C}_\mu$ preserves, $\mathcal{M}$-sized limits, then

1. *Each category $\widehat{\mathscr{C}}_p$ contains $\mathscr{C}_p$ as a reflective subcategory.*
2. *Each functor $\widehat{\mathscr{C}}_\mu : \widehat{\mathscr{C}}_p \to \widehat{\mathscr{C}}_q$ has a left adjoint.*

The category of liftings

Given $\mu : p \to q$ and $\nu : r \to q$, let $\mathsf{Fact}^{\mu}_{\nu}$ be the set of:

- pairs (ϱ, α) of a modality
  $\varrho : p \to r$ and a law $\alpha : \mu \Rightarrow \nu \circ \varrho$



The ∼LSR approach to $(x :^{\mu} A)/\nu$ has one variable $(x :^{\varrho} A)$ for each $(\varrho, \alpha) \in \mathsf{Fact}^{\mu}_{\nu}$, hence semantically the product

$$\big(\Gamma, (x :^{\mu} A)\big)/\nu \;\equiv\; \big(\Gamma/\nu,\; \textstyle\prod_{(\varrho,\alpha)\in\mathsf{Fact}^{\mu}_{\nu}} (x :^{\varrho} A)\big)$$

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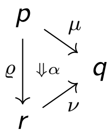

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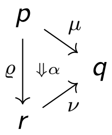

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This ignores the morphisms in $\mathsf{Fact}^{\mu}_{\nu}$! We should instead use

$$\big(\Gamma, (x :^{\mu} A)\big)/\nu \;\equiv\; \big(\Gamma/\nu,\; \lim_{(\varrho,\alpha)\in\mathsf{Fact}^{\mu}_{\nu}} (x :^{\varrho} A)\big)$$

In the "coinductive" $\widehat{\mathscr{C}}$, this defines $\Gamma, (x :^{\mu} A)$ by copatterns.

Conclusion

Concluding remarks:

- Given $\mathscr{C}$, we can interpret MTT in $\widehat{\mathscr{C}}$ to reason about $\mathscr{C}$.
- If we think of $\widehat{\mathscr{C}}$ as a "coherence construction" applied to $\mathscr{C}$, we can say MTT has semantics in functors $\mathscr{C} : \mathcal{M} \to \mathcal{C}at$.
- Right adjoint negative/Fitch-style modalities also lift to $\widehat{\mathscr{C}}$.

Open problems:

- Does it work for homotopy type theory and higher categories?
- Can we weaken the assumption of $\mathcal{M}$-sized limit-preservation?