Lenses and Dialectica Constructions

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TOPOS INSTITUTE

Thanks!

Thank you Paul, Marie and Jules for the invitation today!



Thanks Perdita Stevens, Jeremy Gibbons and James McKinna for information

Thanks Matteo Cappucci, Jon Weinberger and Bruno Gavranović for MRC 'Applied Category Theory' discussions

Lenses? Later

Dialectica construction: An underdog tradition!

earlier: Chu spaces,

Lenses? Later

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2022: MRC Applied Category Theory

Dial and Poly, Niu et al
Dial and Games, Koenig et al
Dial and Lenses, Cappucci et al
Dial and Petri Net Processes, Aten/Bond

Dialectica Interpretation

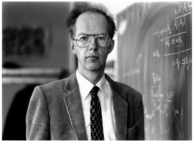
Dialectica Interpretation (Gödel 1958): an interpretation of intuitionistic arithmetic HA in a quantifier-free theory of functionals of finite type **System $\mathbf{T}$**.

Idea: translate every formula A of HA to

$$A^D = \exists u \forall x A_D$$

where A_D is quantifier-free.

Dialectica Interpretation



Application (Gödel 1958): if HA proves A, then System T proves $A_D(t, x)$, where x is a string of variables for functionals of finite type, and t a suitable sequence of terms (not containing x).

Goal: to be as **constructive** as possible, while being able to interpret all of classical Peano arithmetic (Troelstra).

Gödel (1958), *Über eine bisher noch nicht benützte erweiterung des finiten standpunktes.*, Dialectica, 12(3-4):280–287. (Translation in Gödel's Collected Works)

Dialectica interpretation

$A_D(u;x)$ quantifier-free formula defined inductively:

$$
\begin{array}{lcl}
(P)_D & \equiv & P \ (P \text{ atomic}) \\
(A \wedge B)_D(u,v;x,y) & \equiv & A_D(u;x) \wedge B_D(v;y) \\
(A \vee B)_D(u,v,z;x,y) & \equiv & (z=0 \to A_D(u;x)) \wedge (z \neq 0 \to B_D(v;y)) \\
(A \to B)_D(f,F;u,y) & \equiv & A_D(u;Fuy) \to B_D(fu;y) \\
(\exists z A)_D(u,x;z) & \equiv & A_D(u;x) \\
(\forall z A)_D(f;y,z) & \equiv & A_D(fz;y)
\end{array}
$$

Theorem (Dialectica Soundness, Gödel 1958)

Whenever a formula A is provable in Heyting arithmetic then there exists a sequence of closed terms t such that $A_D(t;y)$ is provable in system T. The sequence of terms t and the proof of $A_D(t;y)$ are constructed from the given proof of A in Heyting arithmetic.

Dialectica interpretation

The most complicated clause of the translation is the definition of the translation of the **implication connective** $(A \to B)^D$

$$(A \to B)^D = \exists f, F \forall u, y (A_D(u, F(u, y)) \to B_D(f(u), y)).$$

Intuition: Given a witness u in U for the hypothesis A_D, there exists a function f assigning a witness $f(u)$ to B_D. Moreover, from a counterexample y to the conclusion B_D, we should be able to find a counterexample $F(u, y)$ for the hypothesis A_D.

Feferman et al editors (1986), *Kurt Gödel: Collected Works: Volume II*, Oxford University Press.

Dialectica interpretation

Troelstra (p 226 Collected Works Gödel) from Spector (1962)

$$[\exists u \forall x. A_D(u,x) \to \exists v \forall y. B_D(v,y)] \leftrightarrow^{(i)}$$

$$[\forall u(\forall x A_D(u,x) \to \exists v. \forall y(B_D(v,y))] \leftrightarrow^{(ii)}$$

$$[\forall u \exists v(\forall x. A_D(u,x) \to \forall y B_D(v,y))] \leftrightarrow^{(iii)}$$

$$[\forall u \exists v \forall y(\forall x A_D(u,x) \to B_D(v,y))] \leftrightarrow^{(iv)}$$

$$[\forall u \exists v \forall y \exists x(A_D(u,x) \to B_D(v,y))] \leftrightarrow^{(v)}$$

$$\exists V, X \forall u, y(A_D(u, X(u,y)) \to B_D(V(u), y))$$

where (i) and (iii) are intuitionistic, but (ii) requires **Independence of Premise**, (iv) requires **Markov Principle** and (v) requires two uses of the **axiom of choice**.

Categorical Dialectica Construction

Dialectica category (de Paiva 1988): Given a category C with finite limits, one can build a new category $\mathfrak{Dial}(\mathrm{C})$, whose objects have the form $A = (U, X, \alpha)$ where α is a subobject of $U \times X$ in C; **think** of this object as representing the formula

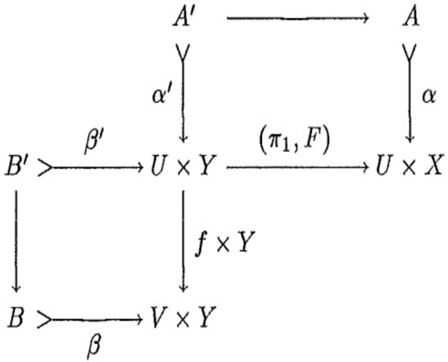
$$\exists u \forall x \alpha(u, x).$$

A map from $\exists u \forall x \alpha(u, x)$ to $\exists v \forall y \beta(v, y)$ can be thought of as a pair $(f : U \to V, F : U \times Y \to X)$ of terms/maps, subject to the entailment condition

$$\alpha(u, F(u, y)) \vdash \beta(f(u), y).$$

(First internalisation of the Dialectica interpretation!)

A drawing or two



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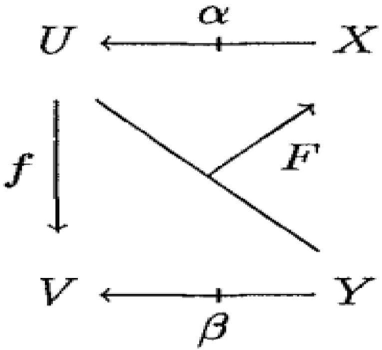

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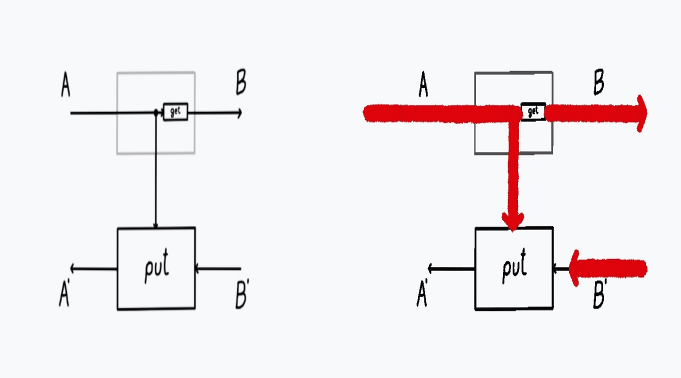
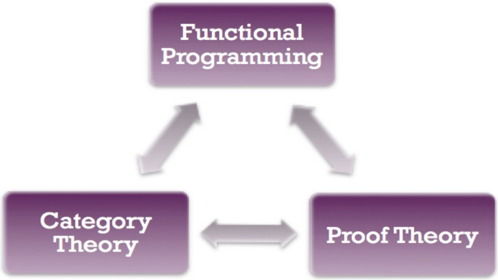


what are we using these maps for?

(Thanks for the animation Bruno Gavranović!
https://www.brunogavranovic.com/posts/2022-02-10-optics-vs-lenses-operationally.html)

Categorical Models

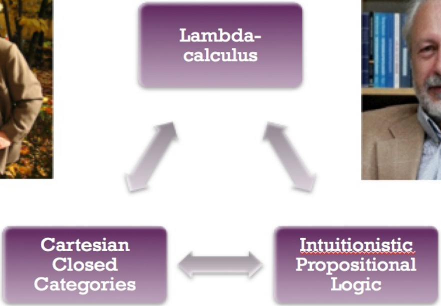


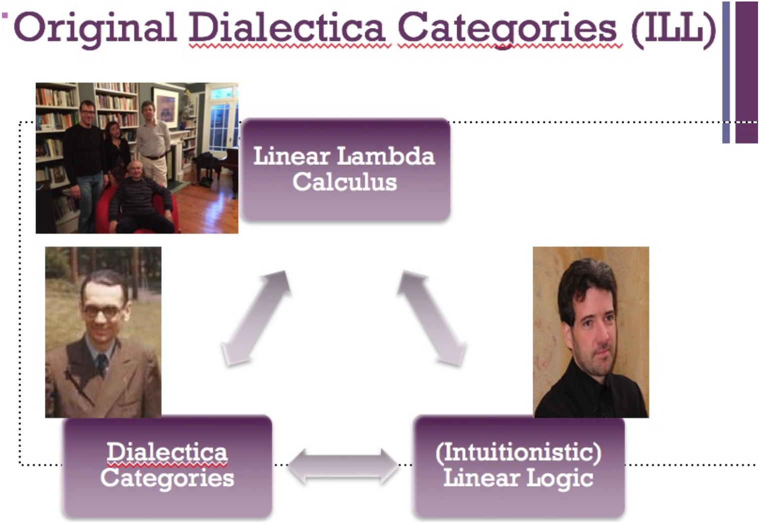
Types are formulae/objects in appropriate category,
Terms/programs are proofs/morphisms in the category,
Logical constructors are 'appropriate' categorical constructions.
Most important: Reduction is proof normalization (**Tait**)
Outcome: Transfer results/tools from logic to CT to CScience

+ Curry-Howard Correspondence

1963

1965





linear cats
LNL adjunctions

Category $\mathfrak{Dial}(C)$

Start with a cat C that is cartesian closed with pullbacks. Then build a new category $\mathfrak{Dial}(C)$.

Objects are relations in C, triples (U, X, α), $\alpha : U \times X \to 2$, so either $u\alpha x$ or not.

Maps are pairs of maps in C. A map from $A = (U, X, \alpha)$ to $B = (V, Y, \beta)$ is a pair of maps in C, $(f : U \to V, F : U \times Y \to X)$ such that a 'semi-adjunction condition' is satisfied: for $u \in U, y \in Y$, $u\alpha F(u, y)$ implies $fu\beta y$. (**Note direction and dependence!**)

Category $\mathfrak{Dial}(C)$

Theorem1: (de Paiva 1987) [Linear structure]

The category $\mathfrak{Dial}$(C) has a symmetric monoidal closed structure (and products, **weak** coproducts), that makes it a model of (exponential-free) **intuitionistic** linear logic.

To model !: Take $!(U, X, \alpha) = (U, X^*, \alpha *)$, where $(-)^*$ is the free commutative monoid in C.

Theorem2: [linear and usual logic together]

There is a **monoidal** comonad ! in $\mathfrak{Dial}$(C) which models exponentials/modalities and recovers Intuitionistic (and via DN Classical) Logic.

Intuition for these objects?

Blass makes the case for thinking of problems in terms of computational complexity. Samuel da Silva and I say you can think of **Kolmogorov-Veloso problems** $\Rightarrow$ applications to Set Theory.
Many other interpretations make sense.
Intuitively an object of $\mathfrak{Dial}(\mathrm{C})$

$$A = (U, X, \alpha)$$

can be seen as representing a problem.
The elements of U are instances of the problem, while the elements of X are possible answers to the problem instances.
The relation α checks whether the answer is correct for that instance of the problem or not.

(Superpower games?)

Examples of objects in $\mathfrak{Dial}(C)$

1. The object $(\mathbb{N}, \mathbb{N}, =)$ where n is related to m iff $n = m$.
2. The object $(\mathbb{N}^{\mathbb{N}}, \mathbb{N}, \alpha)$ where f is α-related to n iff $f(n) = n$.
3. The object $(\mathbb{R}, \mathbb{R}, \leq)$ where r_1 and r_2 are related iff $r_1 \leq r_2$
4. The objects $(2, 2, =)$ and $(2, 2, \neq)$ with usual equality inequality.

The point of Dialectica categories?

A model of Linear Logic, instead of Intuitionistic Logic. (Justifies LL in terms of a traditional proof-theoretic tool and conversely explains the traditional tool in terms of a ‘modern’ linear, resource conscious decomposition.)

A *good* model of Linear Logic: keep the differences that Girard wanted to make. (work with Andrea Schalk on L-valued models of LL).

Justifies claims about Curry-Howard and Harper’s Trinitarism, connections to programming and using CT as syntax guidance.

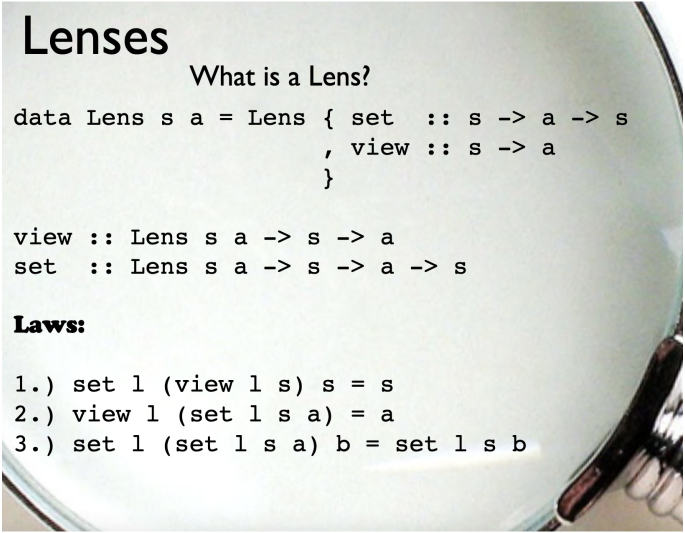
Now: work with Trotta and Spadetto allows us to see where the assumptions in Gödel’s argument (hacks?) are used, based on Hofstra’s explanation.

Lenses

 NO 

YES!

Actually: Kmett's library



http://comonad.com/haskell/Lenses-Folds-and-Traversals-NYC.pdf

Lenses, Forster et al 2005

a solution to the well-known “view update problem"for the case of tree-structured data.

...a language of bi-directional transformations on trees

Each lens must include two functions–one for extracting an abstract view from a concrete one and another for pushing an updated abstract view back into the original concrete view to yield an updated concrete view. We call these the get and putback components, respectively.

Definition: A lens L comprises a partial function from U to U, called the **get** function of L, and a partial function from $U \times U$ to U, called the **putback** function of L.

The ‘identity lens’ copies the concrete view in the get direction and the abstract view in the putback direction. (so identity and composition work as in dialectica)

Lens and Dialectica

Spivak: Generalized Lens Categories via Functors $C^{op} \to Cat$, 2019.

Roughly speaking, a *lens* is bi-directional map $\binom{\text{get}}{\text{put}} : \binom{c}{x} \to \binom{d}{y}$ between pairs; the two parts have the following form:

$$\text{get}: c \to d \qquad \text{and} \qquad \text{put}: c \times y \to x. \tag{1}$$

It is clear that a lens is exactly a(n impoverished) morphism of the Dialectica category $\mathfrak{Dial}(Set)$.

Instead of $\begin{pmatrix} c \\ x \end{pmatrix}$, we write objects horizontally and use capital letters for sets (U, X). A morphism from (U, X) to (V, Y) is a pair of maps $get : U \to V$ and $put : U \times Y \to X$.
(The lens is impoverished because objects do not have an explicit relation between them and morphisms do not have to 'preserve' relations.)

2018: Lenses for Philosophers

Using Spivak's definition is easy to see that there is a forgetful functor from $\mathfrak{Dial}(Set)$ to a category of set-based lenses.

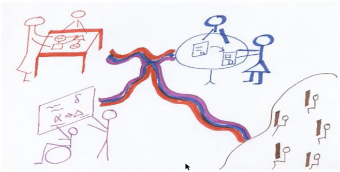
But there are plenty of lenses around: van Laarhoven lenses, delta lenses, monadic lenses, relational lenses, profunctor lenses, etc.

Explains the beginning of Hedges 2018 blog post `https://julesh.com/2018/08/16/lenses-for-philosophers/`.

> *A proof of $\varphi \to \psi$ is a **lens** from proofs of φ to proofs of ψ! Crucially, if we cut together a proof of $\varphi \to \psi$ and a proof of $\psi \to \chi$, the resulting proof of $\varphi \to \chi$ is obtained by ordinary lens composition. But is it reasonable to call them lenses, though?*

BX and Lenses

In practice, models are not independent



Perdita Stevens, Intro to Bidirectionality
`https://www.youtube.com/watch?v=wKwDCi9zN08`

Also Introduction to Bidirectional Transformations, Abou-Saleh, Cheney, Gibbons, McKinna, and Stevens, 2016.
"Pierce and others [26] have spearheaded a fruitful line of work on lenses, programming abstractions for data references supporting 'get' and 'put' operations".

BX and Lenses

Stevens is clear that you use bidirectional transformations to preserve and restore the **consistency** of the systems you're connecting.

In this case having Dialectica spaces instead of lenses seems a more faithful approach, as you have to specify some consistency relation for objects.

Of course one can simplify, forget any explicit consistency relation and deal simply with lenses, which have already (more than one) formidable programming library.

Conclusions

Original Dialectica category
with its symmetric cofree comonad is back in fashion

Lenses of several different stripes are being hotly discussed.
(not only cats, but fibrations, hyperdoctrines, double cats, 2-cats, etc)

Wish I could understand better the **applications*** to see which generalizations make sense to me.

Thank you!

Some applications (see https://github.com/vcvpaiva/DialecticaCategories)

long line of generalizations to doctrines and fibrations: Hyland, Hofstra, Biering, von Glehn and S. Moss. also Trotta, Weinberger, de Paiva *The dependent Godel fibration*, CT2023, Belgium, in two weeks

Trotta, Spadetto, V *Dialectica Principles via Gödel Doctrines*, TCS2023, and J Log&Comp 2022

Dialectica games, J. Koenig et al, JMM2023, but also Winskel LICS23 next week

di Lavore, Leal, de Paiva: *Dialectica Petri nets*, arxiv

da Silva, de Paiva *Kolmogorov-Veloso Problems*, 2021

J. Hedges *Dialectica categories and games with bidding*, 2014 and much other work.

Kerjean, Pédrot *∂ is for Dialectica: typing differentiable programming*. Pédrot's LiCS2014 *A Functional Functional Interpretation*.

Shulman, *The 2-Chu-Dialectica construction and the polycategory of multivariable adjunctions*, TAC 35 (2020)

Budiu, Galenson, Plotkin *The Compiler Forest*, ESOP 2013.

Pradic, Riba *A Dialectica-Like Interpretation of a Linear MSO*

Some References

(see https://github.com/vcvpaiva/DialecticaCategories)

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