Aczel-Mendler Bisimulations in a Regular Category

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Jérémy Dubut
jeremy.dubut@aist.go.jp

National Institute of Advanced Industrial Science and Technology, Tokyo, Japan

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Let's Start Easy:

LTSs, Strong Bisimulations, and Composition

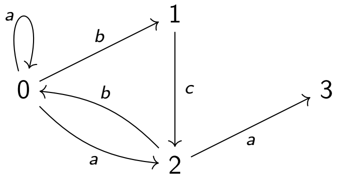
Transition Systems

Labelled Transition System:

A **TS** $T = (Q, \Delta)$ on the alphabet Σ is the following data:

- a set Q (of **states**) and
- a set of **transitions** $\Delta \subseteq Q \times \Sigma \times Q$.

- $\Sigma = \{a, b, c\}$,
- $Q = \{0, 1, 2, 3\}$,
- $\Delta = \{(0, a, 0), (0, b, 1), (0, a, 2), (1, c, 2), (2, b, 0), (2, a, 3)\}$.



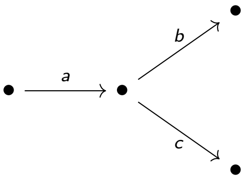
Strong Bisimulations of Transition Systems

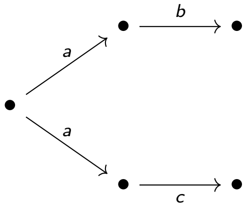
Strong Bisimulations [Park81]:

A **strong bisimulation** between $T_1 = (Q_1, \Delta_1)$ and $T_2 = (Q_2, \Delta_2)$ is a relation $R \subseteq Q_1 \times Q_2$ such that:

(i) if $(q_1, q_2) \in R$ and $(q_1, a, q_1') \in \Delta_1$ then there is $q_2' \in Q_2$ such that $(q_2, a, q_2') \in \Delta_2$ and $(q_1', q_2') \in R$ and

(ii) if $(q_1, q_2) \in R$ and $(q_2, a, q_2') \in \Delta_2$ then there is $q_1' \in Q_1$ such that $(q_1, a, q_1') \in \Delta_1$ and $(q_1', q_2') \in R$.





Strong Bisimulations are Closed under Composition

R_1 strong bisimulation between T_1, T_2 and R_2 strong bisimulation between T_2, T_3

$R_1 ; R_2 = \{(q_1, q_3) \mid \exists q_2. (q_1, q_2) \in R_1 \wedge (q_2, q_3) \in R_2\}$ strong bisimulation between T_1, T_3:

$$\begin{tikzcd}
q_1' & \\
q_1 \arrow[u, "a"] \arrow[r, dashed, "R_1 ; R_2"'] & q_3
\end{tikzcd}$$

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\begin{tikzcd}
q_1' \arrow[r, dash, dashed, "R_1"] & q_2' \arrow[r, dash, dashed, "R_2"] & q_3' \\
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\end{tikzcd}
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$$\begin{tikzcd}
q_1' \arrow[rrr, dash, dashed, "R_1 ; R_2"] & & & q_3' \\
q_1 \arrow[u, "a"] \arrow[rrr, dash, dashed, "R_1 ; R_2"'] & & & q_3 \arrow[u, "a"']
\end{tikzcd}$$

Aczel-Mendler Bisimulations of Coalgebras

Transition systems, as coalgebras

Set of transitions, as functions:

There is a bijection between sets of transitions $\Delta \subseteq Q \times \Sigma \times Q$ and functions of type:

$$\delta : Q \longrightarrow \mathcal{P}(\Sigma \times Q)$$

where $\mathcal{P}(X)$ is the powerset $\{U \mid U \subseteq X\}$.

Coalgebras:

Given an endofunctor $G : \mathcal{C} \longrightarrow \mathcal{C}$, a **coalgebra** is the following data:

- an object $Q \in \mathcal{C}$ and
- a morphism $\delta : Q \longrightarrow G(Q)$ of $\mathcal{C}$.

For LTS: $\mathcal{C} = \mathbf{Set}$, $G = X \mapsto \mathcal{P}(\Sigma \times X)$

Relations in a Category

Subobjects:

There is a preorder on monos with codomain X given by:

$$(u : U \rightarrowtail X) \sqsubseteq (v : V \rightarrowtail X) \quad \Leftrightarrow \quad \exists w : U \rightarrowtail V.\, u = v \cdot w.$$

A subobject of X is an equivalence class of monos $u : U \rightarrowtail X$ modulo $\sqsubseteq \cap \sqsupseteq$.

Ex: in **Set**, subobjects are subsets, and $\sqsubseteq$ is the inclusion

Relations:

A relation R from X to Y is a subobject of $X \times Y$.

Categories with Nice Relations: Regular Categories

Regular Categories:

A regular category is a finitely-complete category with a pullback-stable image factorization. In particular, it means it has a functorial pullback-stable (regular epi, mono)-factorization.

Ex: **Set**, any (quasi)topos, any abelian category, **Stone**, ...

Allegories of Relations:

Given a regular category $\mathcal{C}$, then objects of $\mathcal{C}$ and relations between them form an allegory $\mathbf{Rel}(\mathcal{C})$, i.e.:

- it is a locally ordered 2-category,
- it has an anti-involution $(\_)^\dagger : \mathbf{Rel}(\mathcal{C})^{\mathrm{op}} \to \mathbf{Rel}(\mathcal{C})$,
- local posets are meet-semilattices,
- it satisfies the modular law $(R; S) \cap T \sqsubseteq (R \cap (T; S^\dagger)); S$.

Composition of Relations

Take two relations $m_r : R \rightarrowtail X \times Y$ and $m_s : S \rightarrowtail Y \times Z$, the composition $m_{r;s} : R; S \rightarrowtail X \times Z$ is given by:

$$
\begin{tikzcd}
R \star S \arrow[r, "\mu_2"] \arrow[d, "\mu_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & S \arrow[d, "\pi_1 \cdot m_s"] \\
R \arrow[r, "\pi_2 \cdot m_r"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
R \star S \arrow[rr, "\langle \pi_1 \cdot m_r \cdot \mu_1 , \pi_2 \cdot m_s \cdot \mu_2 \rangle"] \arrow[dr, two heads, "e_{r;s}"'] & & X \times Z \\
& R; S \arrow[ur, tail, "m_{r;s}"'] &
\end{tikzcd}
$$

In **Set**:

- $R \star S = \{(x, y, z) \mid (x, y) \in R \wedge (y, z) \in S\}$,
- $R \star S \to X \times Z$ is given by $(x, y, z) \mapsto (x, z)$, and
- the image $R; S$ is $\{(x, z) \mid \exists y \in Y. (x, y) \in R \wedge (y, z) \in S\}$.

Aczel-Mendler Bisimulations

Aczel-Mendler Bisimulations:

A relation $r : R \rightarrowtail X \times Y$ is an AM-bisimulation from $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$ if there is a morphism $W : R \longrightarrow FR$ (witness) such that:

$$\begin{tikzcd}
 & X \times Y \arrow[r, "\alpha \times \beta"] & F(X) \times F(Y) \\
R \arrow[ur, "r"] \arrow[dr, "W"'] & & \\
 & FR \arrow[r, "Fr"'] & F(X \times Y) \arrow[uu, "\langle F\pi_1, F\pi_2 \rangle"']
\end{tikzcd}$$

In **Set**, for $F : X \mapsto \mathcal{P}(\Sigma \times X)$, AM-bisimulations are strong bisimulations:

- Fix $(x, y) \in R$, and $(a, x') \in \alpha(x)$.
- Commutativity means $(a, x') \in F(\pi_1 \cdot r) \cdot W(x, y)$, that is, there is y' such that $(a, (x', y')) \in W(x, y) \subseteq \Sigma \times R$, and $(x', y') \in R$.
- Commutativity means $(a, y') \in F(\pi_2 \cdot r) \cdot W(x, y) = \beta(y)$.

AM-Bisimulations are Closed under Composition?

Closure under composition:

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$$
\begin{tikzcd}
q_1' & & \\
q_1 \arrow[u, "a"] \arrow[r, dashed, no head, "R_1"'] & q_2 \arrow[r, dashed, no head, "R_2"'] & q_3
\end{tikzcd}
$$

That was a choice of an intermediate state!

Proof

Starting with

$$\begin{tikzcd} & X \times Y \arrow[r, "\alpha \times \beta"] & F(X) \times F(Y) \\ R_1 \arrow[ur, "r_1"] \arrow[dr, "W_1"'] & & \\ & FR_1 \arrow[r, "Fr_1"'] & F(X \times Y) \arrow[uu, "\langle F\pi_1, F\pi_2 \rangle"'] \end{tikzcd} \qquad \begin{tikzcd} & Y \times Z \arrow[r, "\beta \times \gamma"] & F(Y) \times F(Z) \\ R_2 \arrow[ur, "r_2"] \arrow[dr, "W_2"'] & & \\ & FR_2 \arrow[r, "Fr_2"'] & F(Y \times Z) \arrow[uu, "\langle F\pi_1, F\pi_2 \rangle"'] \end{tikzcd}$$

we want W such that

$$\begin{tikzcd} & X \times Z \arrow[r, "\alpha \times \gamma"] & F(X) \times F(Z) \\ R_1 ; R_2 \arrow[ur, "r_1 ; r_2"] \arrow[dr, "W"'] & & \\ & F(R_1 ; R_2) \arrow[r, "F(r_1 ; r_2)"'] & F(X \times Z) \arrow[uu, "\langle F\pi_1, F\pi_2 \rangle"'] \end{tikzcd}$$

Proof

By definition of the composition

$$\begin{tikzcd}
R_1 \star R_2 \arrow[rr, "{\langle \pi_1 \cdot r_1 \cdot \mu_1, \pi_2 \cdot r_2 \cdot \mu_2 \rangle}"] \arrow[dr, two heads, "e_{r_1;r_2}"'] & & X \times Z \\
& R_1 ; R_2 \arrow[ur, tail, "r_1 ; r_2"'] &
\end{tikzcd}$$

Proof

By definition of the composition and the regular axiom of choice

$$\begin{tikzcd}
R_1 \star R_2 \arrow[rr, "\langle \pi_1 \cdot r_1 \cdot \mu_1, \pi_2 \cdot r_2 \cdot \mu_2 \rangle"] \arrow[dr, two heads, "e_{r_1 ; r_2}"] & & X \times Z \\
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Proof

By definition of the composition and the regular axiom of choice

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\end{tikzcd}$$

That was a choice of an intermediate state!

Proof

By definition of the composition, this is a (weak) pullback

$$\begin{tikzcd}
R_1 \star R_2 \arrow[r, "\mu_2"] \arrow[d, "\mu_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & R_2 \arrow[d, "\pi_1 \cdot r_2"] \\
R_1 \arrow[r, "\pi_2 \cdot r_1"'] & Y
\end{tikzcd}$$

Proof

By preservation of weak pullback, this is a weak pullback

$$\begin{tikzcd}
F(R_1 \star R_2) \arrow[r, "F\mu_2"] \arrow[d, "F\mu_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & FR_2 \arrow[d, "F(\pi_1 \cdot r_2)"] \\
FR_1 \arrow[r, "F(\pi_2 \cdot r_1)"'] & FY
\end{tikzcd}$$

Proof

Putting everything together, the following commutes

$$
\begin{tikzcd}
R_1 ; R_2 \arrow[drr, bend left, "W_2 \cdot \mu_2 \cdot s"] \arrow[ddr, bend right, "W_1 \cdot \mu_1 \cdot s"'] & & \\
& F(R_1 \star R_2) \arrow[r, "F\mu_2"] \arrow[d, "F\mu_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & FR_2 \arrow[d, "F(\pi_1 \cdot r_2)"] \\
& FR_1 \arrow[r, "F(\pi_2 \cdot r_1)"'] & FY
\end{tikzcd}
$$

Proof

Then there is ϕ making the triangles commute

$$
\begin{tikzcd}
R_1 ; R_2 \arrow[dr, dotted, "\phi"] \arrow[ddr, bend right, "W_1 \cdot \mu_1 \cdot s"'] \arrow[drr, bend left, "W_2 \cdot \mu_2 \cdot s"] & & \\
& F(R_1 \star R_2) \arrow[r, "F\mu_2"] \arrow[d, "F\mu_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & FR_2 \arrow[d, "F(\pi_1 \cdot r_2)"] \\
& FR_1 \arrow[r, "F(\pi_2 \cdot r_1)"'] & FY
\end{tikzcd}
$$

Proof

Choosing $W = F(e_{r1;r2}) \cdot \phi$ gives what we want:

$$\begin{tikzcd}
 & X \times Z \arrow[r, "\alpha \times \gamma"] & F(X) \times F(Z) \\
R_1; R_2 \arrow[ur, "r_1; r_2"] \arrow[dr, "W"'] & & \\
 & F(R_1; R_2) \arrow[r, "F(r_1; r_2)"'] & F(X \times Z) \arrow[uu, "\langle F\pi_1, F\pi_2 \rangle"']
\end{tikzcd}$$

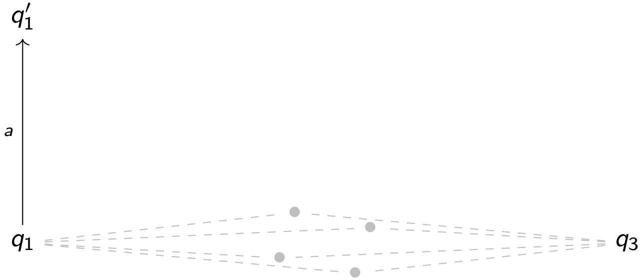
From Picking to Collecting

Regular AM-Bisimulations

Picking vs Collecting

$$\begin{tikzcd}
q_1' & & \\
q_1 \arrow[u, "a"] \arrow[rr, dashed, "R_1 ; R_2"'] & & q_3
\end{tikzcd}$$

Picking vs Collecting



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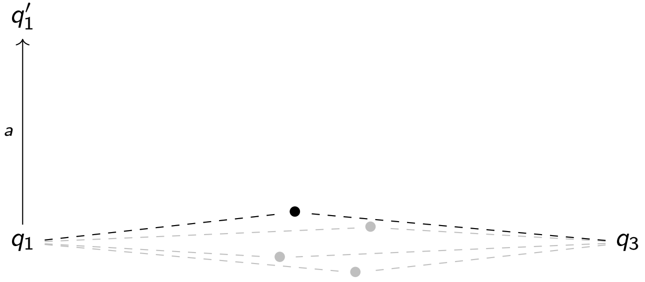
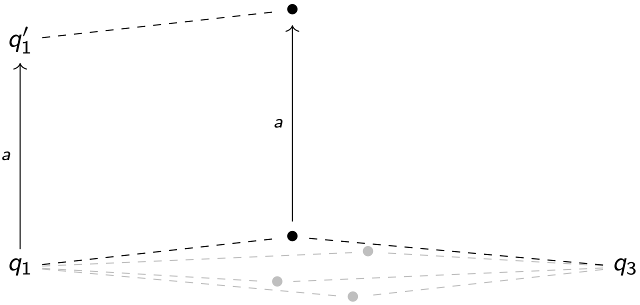


Pick one

Picking vs Collecting

Make the proof for this one



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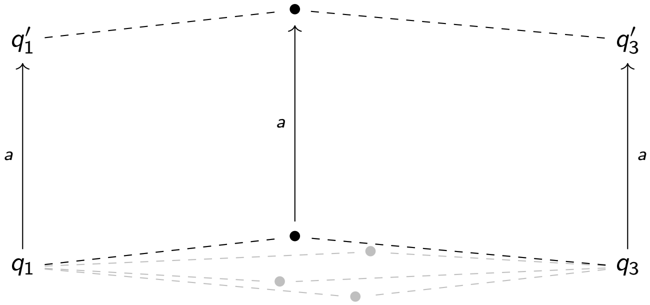

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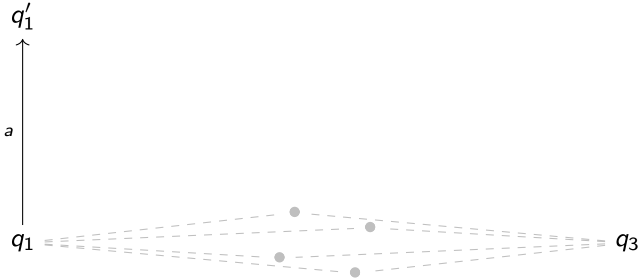
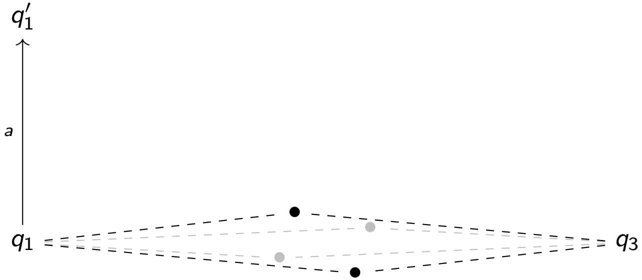
Picking vs Collecting

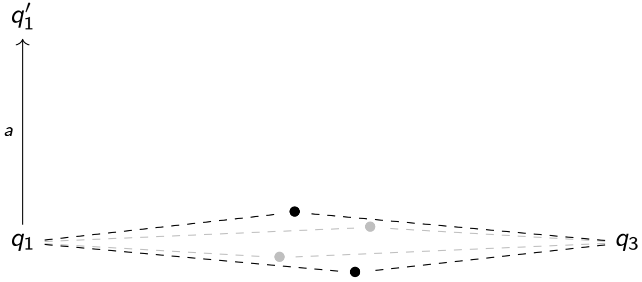


Picking vs Collecting



Collect many

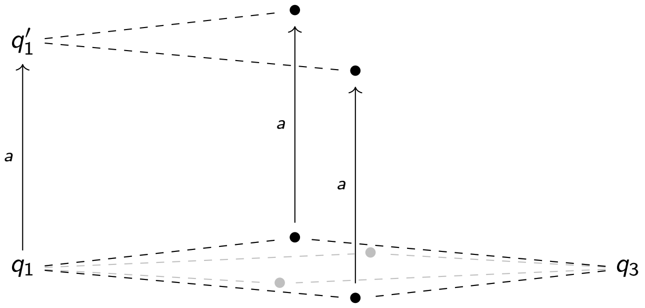
Picking vs Collecting



Collect many
(Make sure there is at least one)

Picking vs Collecting

Make the proof for all of them



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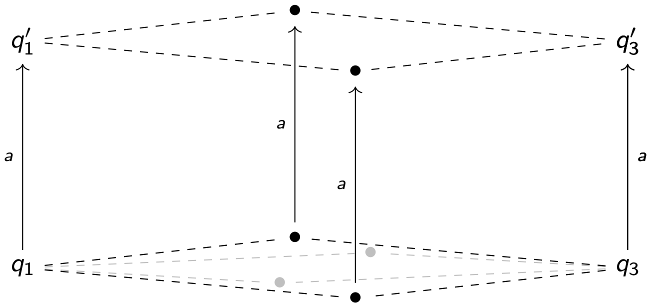

Collect many
(Make sure there is at least one)

How to do that, abstractly?

Instead of building a *witness function*:

$$W : R \longrightarrow FR$$

build a *witness relation*:

$$w \ : \ W \rightarrowtail R \times FR$$

Regular AM-Bisimulations

Regular Aczel-Mendler Bisimulations:

A relation $r : R \rightarrowtail X \times Y$ is a regular AM-bisimulation from $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$ if there is a relation $w : W \rightarrowtail FR \times R$ (witness) such that $\pi_2 \cdot w$ is a regular epi and :

$$
\begin{tikzcd}
& R \arrow[r, "r"] & X \times Y \arrow[dr, "\alpha \times \beta"] & \\
W \arrow[ur, "\pi_2 \circ w"] \arrow[dr, "\pi_1 \circ w"'] & & & F(X) \times F(Y) \\
& FR \arrow[r, "Fr"'] & F(X \times Y) \arrow[ur, "\langle F\pi_1, F\pi_2 \rangle"'] &
\end{tikzcd}
$$

Basic Properties

Regular AM-Bisimulations Form a Dagger 2-Poset:

- Diagonals are regular AM-bisimulations.
- Regular AM-bisimulations are closed under inverse.
- When F covers pullbacks, regular AM-bisimulations are closed under composition.

Coincidence under the Axiom of Choice:

When $\mathcal{C}$ has the regular axiom of choice, then regular AM-bisimulations coincide with AM-bisimulations.

Relationship with Other Coalgebraic Bisimulations:

- Regular AM-bisimulations coincide with Hermida-Jacobs bisimulations.
- When F covers pullbacks, then behavioral equivalences are AM-bisimulations.
- When $\mathcal{C}$ has pushouts, every regular AM-bisimulation is included in a behavioral equivalence.

Example: Vietoris Bisimulations in **Stone**

Objects: Stone spaces, i.e., compact totally disconnected spaces

Morphisms: continuous functions

This is a regular category with pushouts

Subobjects: **closed** subsets

Vietoris functor $\mathcal{V}$: $X \mapsto$ set of closed subsets of X with a suitable topology

This endofunctor covers pullbacks (but do not preserve weak-pullbacks!)

[Bezhanishvili et al.'10]

Fix a Stone space A. Descriptive models coincide with $\mathcal{V}(\_) \times A$-coalgebras.
Vietoris bisimulations coincide with HJ-bisimulations (so with regular AM too), but not with plain AM-bisimulations.

Counter-example from [Bezhanishvili et al.'10]

We want to construct two coalgebras $X \mapsto \mathcal{V}(X) \times \mathcal{P}(\mathbb{N} \times \{+, -\})$ in **Stone**

$$\begin{array}{cc} \vdots & \vdots \\ 2n+1 & \circ \\ 2n & \circ \\ \vdots & \vdots \\ 2 & \circ \\ 1 & \circ \\ 0 & \circ \end{array}$$

Counter-example from [Bezhanishvili et al.'10]

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∞	$\circ$
$\vdots$	$\vdots$
$2n+1$	$\circ$
$2n$	$\circ$
$\vdots$	$\vdots$
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1	$\circ$
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$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2n+1$	$\circ$	$\circ$	$\circ$
$2n$	$\circ$	$\circ$	$\circ$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
2	$\circ$	$\circ$	$\circ$
1	$\circ$	$\circ$	$\circ$
0	$\circ$	$\circ$	$\circ$

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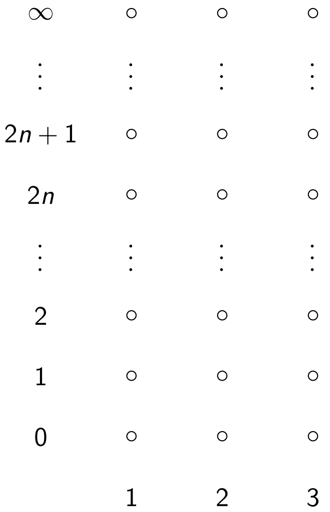

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$$
\begin{array}{cccc|cccc}
\infty & \circ & \circ & \circ & \circ & \circ & \circ & \infty \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
2n+1 & \circ & \circ & \circ & \circ & \circ & \circ & 2n+1 \\
2n & \circ & \circ & \circ & \circ & \circ & \circ & 2n \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
2 & \circ & \circ & \circ & \circ & \circ & \circ & 2 \\
1 & \circ & \circ & \circ & \circ & \circ & \circ & 1 \\
0 & \circ & \circ & \circ & \circ & \circ & \circ & 0 \\
 & 1 & 2 & 3 & 1 & 2 & 3 &
\end{array}
$$

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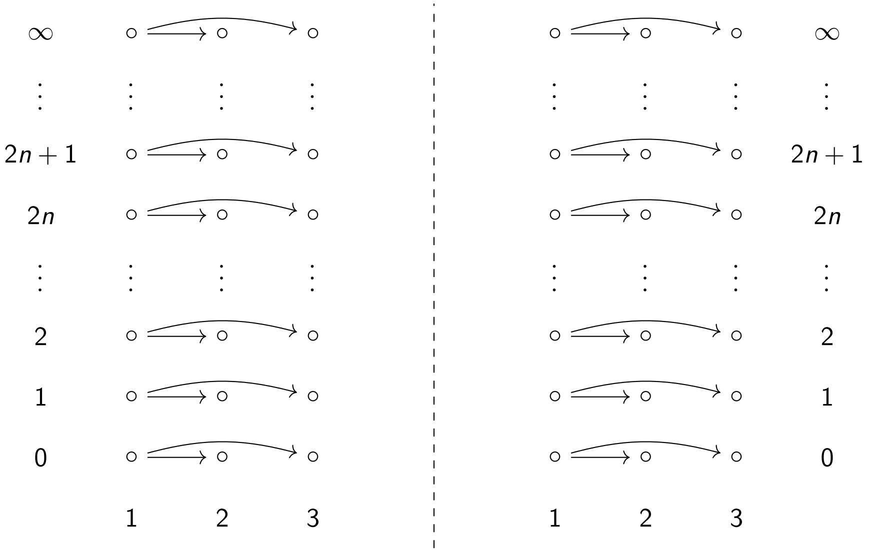

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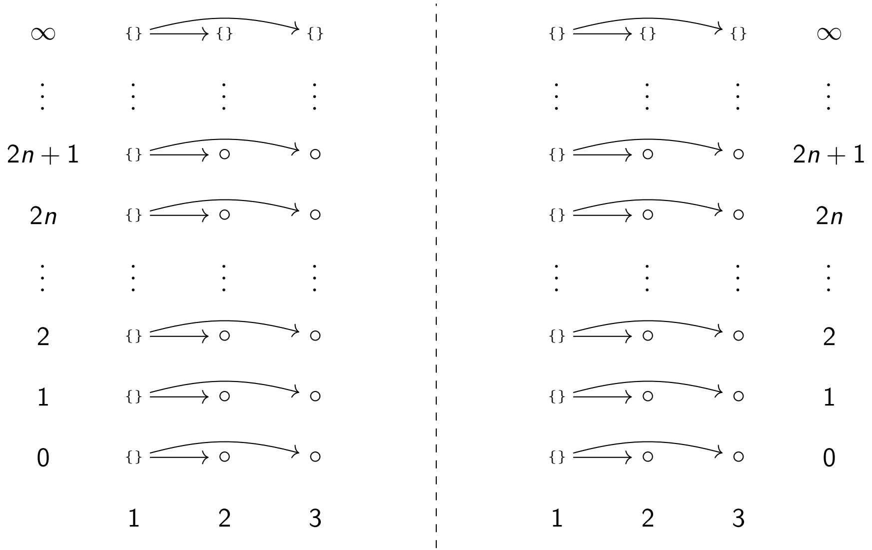

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$$
\begin{tikzcd}
\infty & \{\} \arrow[r] \arrow[rr, bend left] & \{\} & \{\} \\
\vdots & \vdots & \vdots & \vdots \\
2n+1 & \{\} \arrow[r] \arrow[rr, bend left] & \{(\mathbf{2}n+\mathbf{1})+\} & \{(\mathbf{2}n+\mathbf{1})-\} \\
2n & \{\} \arrow[r] \arrow[rr, bend left] & \circ & \circ \\
\vdots & \vdots & \vdots & \vdots \\
2 & \{\} \arrow[r] \arrow[rr, bend left] & \circ & \circ \\
1 & \{\} \arrow[r] \arrow[rr, bend left] & \{\mathbf{1}+\} & \{\mathbf{1}-\} \\
0 & \{\} \arrow[r] \arrow[rr, bend left] & \circ & \circ \\
 & 1 & 2 & 3
\end{tikzcd}
$$

$$
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\{\} \arrow[r] \arrow[rr, bend left] & \{\} & \{\} & \infty \\
\vdots & \vdots & \vdots & \vdots \\
\{\} \arrow[r] \arrow[r, bend left] & \{(\mathbf{2}n+\mathbf{1})+\} & \{(\mathbf{2}n+\mathbf{1})-\} & 2n+1 \\
\{\} \arrow[r] \arrow[rr, bend left] & \circ & \circ & 2n \\
\vdots & \vdots & \vdots & \vdots \\
\{\} \arrow[r] \arrow[rr, bend left] & \circ & \circ & 2 \\
\{\} \arrow[r] \arrow[rr, bend left] & \{\mathbf{1}+\} & \{\mathbf{1}-\} & 1 \\
\{\} \arrow[r] \arrow[rr, bend left] & \circ & \circ & 0 \\
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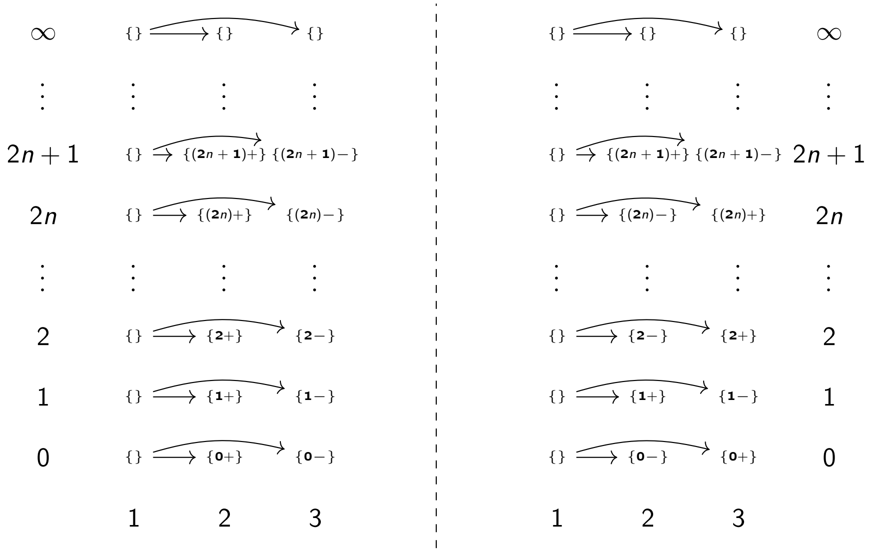

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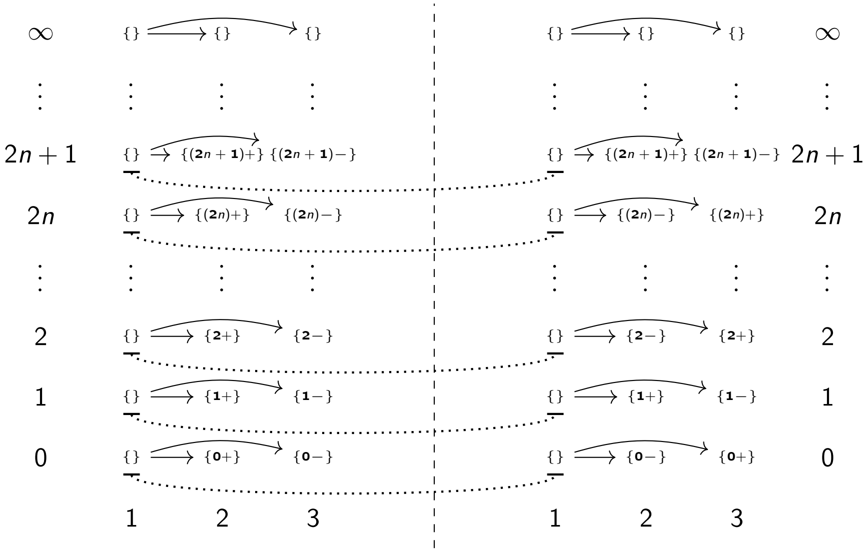

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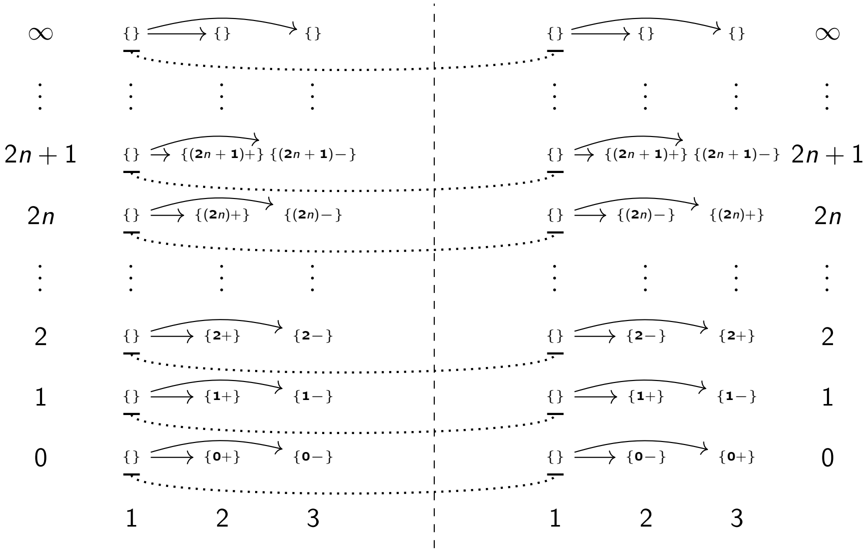

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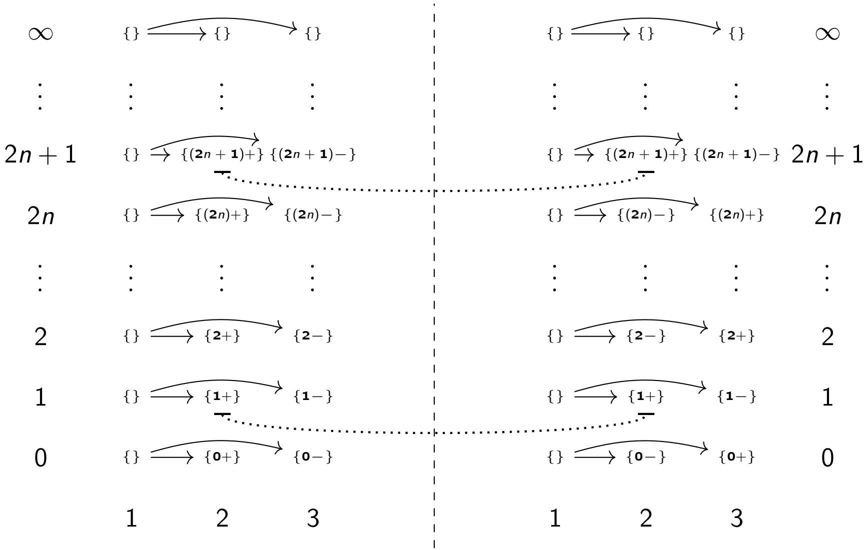

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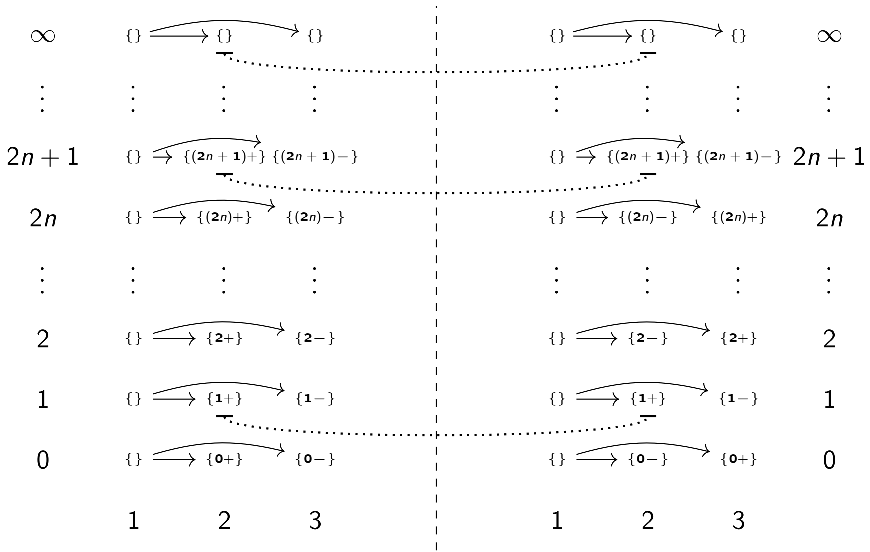

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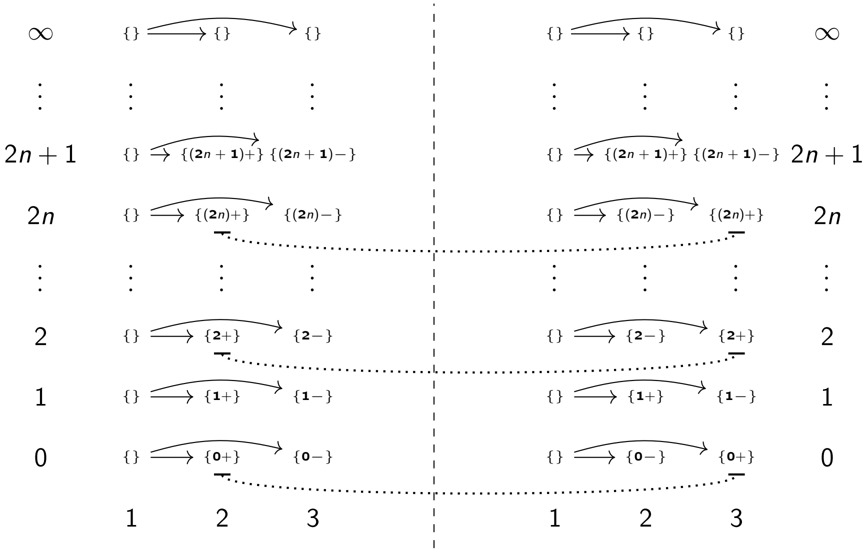

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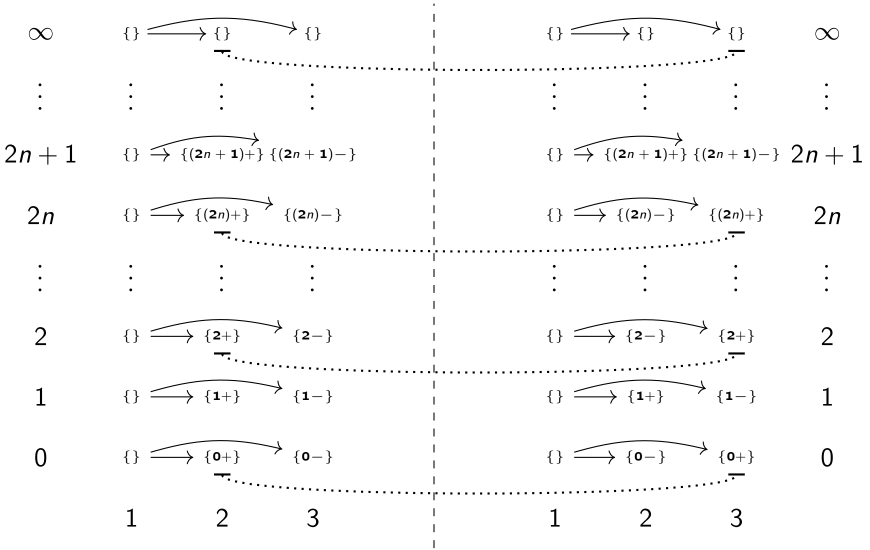

Counter-example from [Bezhanishvili et al.'10]

Regular AM $\neq$ AM

The following closed relation:

$$\begin{aligned} R = \{(i_2, i_2), (i_3, i_3) \mid i \in \mathbb{N} \text{ odd}\} &\cup \{(i_2, i_3), (i_3, i_2) \mid i \in \mathbb{N} \text{ even}\} \\ &\cup \{(i_1, i_1) \mid i \in \mathbb{N} \cup \{\infty\}\} \\ &\cup \{(\infty_j, \infty_k) \mid j, k \in \{2, 3\}\} \end{aligned}$$

is a regular AM-bisimulation but not an AM-bisimulation.

Proof:

The following is a witness closed relation $W \subseteq R \times (\mathcal{V}(R) \times \mathcal{P}(\mathbb{N} \times \{+, -\}))$:

$$\begin{aligned} W = \{((i_1, i_1), \{(i_2, i_2), (i_3, i_3)\}, \{\}) \mid i \in \mathbb{N} \text{ odd}\} \\ \cup \{((i_1, i_1), \{(i_2, i_3), (i_3, i_2)\}, \{\}) \mid i \in \mathbb{N} \text{ even}\} \\ \cup \{((\infty_1, \infty_1), \{(\infty_2, \infty_2), (\infty_3, \infty_3)\}, \{\})\} \\ \cup \{((\infty_1, \infty_1), \{(\infty_2, \infty_3), (\infty_3, \infty_2)\}, \{\})\} \\ \cup \{((i_j, i_k), \varnothing, \lambda(i_j)) \mid i \in \mathbb{N} \cup \{\infty\} \wedge (i_j, i_k) \in R\} \end{aligned}$$

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(\infty_1, \infty_1) \text{ has two witnesses } \Big\{ &\cup \{((\infty_1, \infty_1), \{(\infty_2, \infty_2), (\infty_3, \infty_3)\}, \{\})\} \\
&\cup \{((\infty_1, \infty_1), \{(\infty_2, \infty_3), (\infty_3, \infty_2)\}, \{\})\} \\
&\cup \{((i_j, i_k), \varnothing, \lambda(i_j)) \mid i \in \mathbb{N} \cup \{\infty\} \wedge (i_j, i_k) \in R\}
\end{aligned}
$$

The Special Case of Toposes

Toposes, as Relation Classifiers

Topos:

A topos is a finitely complete category $\mathcal{C}$ with **power objects**, that is, for every object X, there is a mono $\in_X : E_X \rightarrowtail X \times \mathcal{P}X$ such that for every relation $r : R \rightarrowtail X \times Y$ there is a unique morphism $\xi_r : Y \longrightarrow \mathcal{P}X$ such that there is a pullback of the form:

$$\begin{tikzcd} R \arrow[rr, "\theta_r"] \arrow[d, tail, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & & E_X \arrow[d, tail, "\in_X"] \\ X \times Y \arrow[rr, "\mathrm{id} \times \xi_r"'] & & X \times \mathcal{P}X \end{tikzcd}$$

In **Set**: $\mathcal{P}$ = power set, $E_X = \{(x, U) \mid x \in U\}$

The subobject classifier is $\Omega = \mathcal{P}\mathbf{1}$ and $\mathcal{P}X = \Omega^X$

This formulation implies cartesian closure

Folklore and More

Folklore:

$\mathcal{P}$ is a commutative monad whose Kleisli category is isomorphic to the category of relations of $\mathcal{C}$.

[Goy et al'21]

- For every endofunctor F of a topos $\mathcal{C}$ and object X of $\mathcal{C}$, there is a canonical morphism
$$\sigma_{F,X}\colon\ F\mathcal{P}X \to \mathcal{P}FX.$$
- When F preserves weak pullbacks and epis, this is a natural transformation.
- If F is additonally a monad whose multiplication is weak cartesian, σ_F is a weak distributive law.
- If additionally the unit is also weak cartesian, then σ_F is a distributive law.
- In particular, for any non-trivial topos, $\sigma_{\mathcal{P}}$ is a weak distributive law but not a strict one.

A Nicer Formulation of Regular AM-Bisimulations

Toposal Aczel-Mendler Bisimulations:

A relation $r : R \rightarrowtail X \times Y$ is a toposal AM-bisimulation from $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$ if there is a morphism $W : R \longrightarrow \mathcal{P}FR$ (witness) such that:

$$\begin{tikzcd}
 & X \times Y \arrow[r, "\alpha \times \beta"] & F(X) \times F(Y) \arrow[dr, "\eta_{F(X)} \times \eta_{F(Y)}"] & \\
R \arrow[ur, "r"] \arrow[dr, "W"'] & & & \mathcal{P}F(X) \times \mathcal{P}F(Y) \\
 & \mathcal{P}FR \arrow[r, "\mathcal{P}Fr"'] & \mathcal{P}F(X \times Y) \arrow[ur, "\langle \mathcal{P}F\pi_1, \mathcal{P}F\pi_2 \rangle"'] &
\end{tikzcd}$$

Basically, F-toposal-AM $= \mathcal{P}F$-AM

Toposal = Regular

In a topos, toposal AM-bisimulations coincide with regular AM-bisimulations.

Conclusion

- In this talk:
  - Plain AM-bisimulations work only with the axiom of choice.
  - Replacing witness functions by relations $\rightarrow$ regular AM-bisimulations
  - They work without axiom of choice:
    - closure under composition,
    - coincidence with HJ-bisimulations, behavioral equivalences.
  - They are reworded nicely in toposes.
- Not in this talk, but in the paper:
  - Allegory maps that are (toposal) AM-bisimulations are ($\mathcal{P}$)F-coalgebra homomorphisms.
  - Everything can be done for simulations too.
  - More examples (toposes for name-passing, weighted systems in categories of modules)
- Future work:
  - Relation with the $\neg\neg$-closure.
  - Regular AM-bisimulations for continuous probabilistic systems?
  - Regular AM-bisimulations in realizability toposes?