A tour on ecumenical systems – CALCO 2023

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Motivation I – What is a proof?

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Intuitionistic mathematician: but $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$ or $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$??? 😕

Motivation I – What is a proof?

CHAPTER **10**

THE GELFOND-SCHNEIDER THEOREM

1. Hilbert's seventh problem. In 1900 David Hilbert announced a list of twenty-three outstanding unsolved problems. The seventh problem was settled by the publication of the following result in 1934 by A. O. Gelfond, which was followed by an independent proof by Th. Schneider in 1935.

THEOREM 10.1. *If α and β are algebraic numbers with $\alpha \neq 0$, $\alpha \neq 1$, and if β is not a real rational number, then any value of α^β is transcendental.*

Remarks. The hypothesis that "β is not a real rational number" is usually stated in the form "β is irrational." Our wording is an attempt to avoid the suggestion that β must be a real number. Such a number as $\beta = 2 + 3i$, sometimes called a "complex rational number," satisfies the hypotheses of the theorem. Thus the theorem establishes the transcendence of such numbers as 2^i and $2^{\sqrt{2}}$. In general, $\alpha^\beta = \exp\{\beta \log \alpha\}$ is multiple-valued, and this is the reason for the phrase "any value of" in the statement of Theorem 10.1. One value of $i^{-2i} = \exp\{-2i \log i\}$ is e^π, and so this is transcendental according to the theorem.

Before proceeding to the proof of Theorem 10.1, we state an alternative form of the result.

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Motivation I – What is a proof?

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Schneider theorem, and they will be given with proofs in the next section.

LEMMA 10.3. *Consider a determinant with the non-zero element ρ_j^a in the j-th row and $1 + a$-th column, with $j = 1, 2, \cdots, t$ and $a = 0, 1, \cdots, t - 1$. This is called a Vandermonde determinant, and it vanishes if and only if $\rho_j = \rho_k$ for some distinct pair of subscripts j, k.*

This can be found in J. V. Uspensky, *Theory of Equations*, McGraw-Hill, p. 214. The next four lemmas are in Harry Pollard, *The Theory of Algebraic Numbers*, John Wiley, p. 53, p. 60, pp. 63–66, p. 72.

LEMMA 10.4. *Let α and β be algebraic numbers in a field K of degree h over the rationals. If the conjugates of α for K are $\alpha = \alpha_1, \alpha_2, \cdots, \alpha_h$ and for β are $\beta = \beta_1, \beta_2, \cdots, \beta_h$, then the conjugates of $\alpha\beta$ and $\alpha + \beta$ are $\alpha_1\beta_1, \cdots, \alpha_h\beta_h$ and $\alpha_1 + \beta_1, \cdots, \alpha_h + \beta_h$.*

LEMMA 10.5. *If α is an algebraic number, then there is a positive rational integer r such that $r\alpha$ is an algebraic integer.*

LEMMA 10.6. *If K is an algebraic number field of degree h over the rationals, then there exist integers $\beta_1, \beta_2, \cdots, \beta_h$ in K such that every integer in K is expressible uniquely as a linear combination $g_1\beta_1 + \cdots + g_h\beta_h$ with rational integral coefficients. The numbers β_j are called an integral basis for K, and the discriminant of such a basis is a non-zero rational integer.*

LEMMA 10.7. *If α is an algebraic number in a field K of degree h over the rationals, then the norm $N(\alpha)$, defined as the product of α and its conjugates, satisfies the relation $N(\alpha\beta) = N(\alpha) \cdot N(\beta)$. Also $N(\alpha) = 0$ if and only if $\alpha = 0$. If α is an algebraic integer, then $N(\alpha)$ is a rational integer. If α is rational, then $N(\alpha) = \alpha^h$.*

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Motivation I – What is a proof?

Sec. 3 TWO LEMMAS 137

Finally, from complex variable theory we need the concept of entire function, i.e., a function that is analytic in the whole complex plane, and Cauchy's residue theorem. These ideas can be found, for example, in K. Knopp's *Theory of Functions*, vol. I, Dover, p. 112ff. and p. 130.

3. Two lemmas. LEMMA 10.8. *Consider the* m *equations in* n *unknowns*

(10.1)

$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n = 0, \qquad k = 1, 2, \cdots, m,$$

with rational integral coefficients a_{ij}, *and with* $0 < m < n$. *Let the positive integer* A *be an upper bound of the absolute values of all coefficients; thus* $A \geqq |a_{ij}|$ *for all* i *and* j. *Then there is a non-trivial solution* $x_1, x_2, \cdots, x_n$ *in rational integers of equations* (10.1) *such that*

$$|x_j| < 1 + (nA)^{m/(n-m)}, \qquad j = 1, 2, \cdots, n.$$

Proof. Write y_k for $a_{k1}x_1 + \cdots + a_{kn}x_n$ so that to each point $x = (x_1, x_2, \cdots, x_n)$ there corresponds a point $y = (y_1, y_2, \cdots, y_m)$. A point such as x is said to be a *lattice point* if its coordinates x_j are rational integers. If x is a lattice point, then the corresponding point y is also a lattice point because the a_{ij} are rational integers. Let q be any positive integer. Let x range over the $(2q + 1)^n$ lattice points inside or on the n-dimensional cube defined by $|x_j| \leqq q$ for all subscripts j. Then the corresponding values of y_k satisfy

$$|y_k| = \left| \sum_{j=1}^{n} a_{kj}x_j \right| \leqq \sum_{j=1}^{n} |a_{kj}| \cdot |x_j| \leqq \sum_{j=1}^{n} Aq = nAq.$$

Thus, as x ranges over the $(2q + 1)^n$ lattice points as indicated, the corresponding lattice points y have coordinates y_k which are integers among the $2nAq + 1$

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Motivation I – What is a proof?

LEMMA 10.9. *Consider the p equations in q unknowns*

(10.4)

$$\alpha_{k1}\xi_1 + \alpha_{k2}\xi_2 + \cdots + \alpha_{kq}\xi_q = 0, \qquad k = 1, 2, \cdots, p,$$

with coefficients α_{ij} which are integers in an algebraic number field K of finite degree. Assume that $0 < p < q$. Let $A \geqq 1$ be an upper bound for the absolute values of the coefficients and their conjugates for K, thus $A \geqq \| \alpha_{ij} \|$ for all i and j. Then there exists a positive constant c depending on the field K but independent of α_{ij}, p, and q, such that the equations (10.4) have a non-trivial solution $\xi_1, \xi_2, \cdots, \xi_q$ in integers of the field K satisfying

$$\| \xi_k \| < c + c(cqA)^{p/(q-p)}, \qquad k = 1, 2, \cdots, p.$$

Proof. Let h be the degree of K over the field of rational numbers, and let $\beta_1, \beta_2, \cdots, \beta_h$ be an integral basis for the field. If α is any integer of K, then by Lemma 10 6 we can express α uniquely as a linear combination of the integral basis,

$$\alpha = g_1\beta_1 + g_2\beta_2 + \cdots + g_h\beta_h,$$

with rational integral coefficients g_j. Denote the conjugates of α for K by $\alpha = \alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(h)}$, and similarly for the β_j. Taking conjugates in the last equation, by Lemma 10.4 we get

$$\alpha^{(i)} = g_1\beta_1^{(i)} + g_2\beta_2^{(i)} + \cdots + g_h\beta_h^{(i)}, \qquad i = 1, 2, \cdots, h.$$

The determinant $|\beta_j^{(i)}|$ is the discriminant of the basis, and it is not zero by Lemma 10.6. Hence we can solve these equations for the g_j as linear combinations of the $\alpha^{(i)}$, with coefficients dependent only on the basis. Taking absolute values throughout these solutions, we can write

(10.5) $$|g_j| < c_1 \| \alpha \|, \qquad j = 1, 2, \cdots, h,$$

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Motivation I – What is a proof?

$$
\begin{aligned}
|\zeta| &< |\log \alpha|^{-p} \cdot \frac{p}{q} \cdot c_8^p p^{p(3-m)/2} \cdot \frac{2q}{p} \\
&< \{2c_8 |\log \alpha|^{-1}\}^p p^{p(3-m)/2} \\
&= c_9^p p^{p(3-m)/2}.
\end{aligned}
$$

With this estimate for $|\zeta|$, and that of Lemma 10.12 for its conjugates, we write, by (10.10),

$$|N(\zeta)| < c_9^p p^{p(3-m)2}(c^p p^p)^{h-1} = (c_9 c^{h-1})^p p^{-p} = c_0^p p^{-p},$$

where $c_0 = c_9 c^{h-1}$. This and Lemma 10.11 imply that

$$c_0^p p^{-p} > C^{-p}, \qquad Cc_0 > p,$$

for some positive constants independent of n and p. But this is a contradiction, because $p \geqq n$, and we can choose n arbitrarily large.

Notes on Chapter 10

The special case of Theorem 10.1 for any imaginary quadratic irrational β was established by A. O. Gelfond, *Compt. Rend. Acad. Sci. Paris*, **189** (1929), 1224–1226. The original papers establishing Theorem 10.1 are: A. O. Gelfond, *Doklady Akad. Nauk S.S.S.R.*, **2** (1934), 1–6; Th. Schneider, *J. reine angew. Math.*, **172** (1935), 65–69. The American Mathematical Society has provided an English translation (Translation Number 65) of an advanced expository paper by A. O. Gelfond, *The approximation of algebraic numbers by algebraic numbers and the theory of transcendental numbers*, *Uspehi Mat. Nauk* (*N.S.*), **4**, no. 4 (32), 19–49 (1949). There is an exposition of Gelfond's proof by E. Hille, *Amer. Math. Monthly*, **49** (1942), 654–661.

The proof of Theorem 10.1 given here is based on a simplification of Gelfond's proof by C. L. Siegel, *Transcendental Numbers*, Princeton, pp. 80–83.

Although the methods of Chapters 9 and 10 establish the transcendence of wide classes of numbers, there are many unsolved prob-

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Motivation II – What is ecumenism?

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- What (really) are ecumenical systems?
- What are they good for?
- Why should anyone be interested in ecumenical systems?
- What is the real motivation behind the definition and development of ecumenical systems?

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Prawitz: what makes a connective classical or intuitionistic?

Logical inferentialism:

- the meaning of the logical constants can be specified by the rules that determine their correct use;
- proof-theoretical requirements on admissible logical rules: harmony and separability;
- pure logical systems: negation is not used in premises.

- IL: if what you mean by $(A \vee B)$ is $\neg(\neg A \wedge \neg B)$, then I can accept the validity of $(A \vee \neg A)$!

Logical motivation (dialogue by Luiz Carlos)

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- CL: but I do not mean $\neg(\neg A \wedge \neg\neg A)$ by $(A \vee \neg A)$. One must distinguish the excluded-middle from the the principle of non-contradiction. When I say that Goldbach's conjecture is either true or false, I am not saying that it would be contradictory to assert that it is not true and that it is not the case that it is not true!

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- IL: but you must realize that, at the end of the day, you just have one logical operator, like the Quine's dagger (a.k.a. NOR).
- CL: But this is not at all true! The fact that we can define one operator in terms of other operators does not imply that we don't have different operators!

  It is true that we can prove $\vdash (A \vee_c B) \leftrightarrow \neg(\neg A \wedge \neg B)$ in the ecumenical system, but this does not mean that we don't have three different operators: $\neg$, $\vee_c$ and $\wedge$.

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if $x + y = 2z$ then $x \geq z$ or $y \geq z$.

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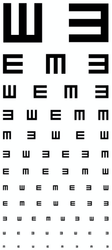

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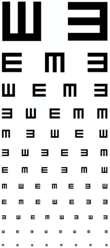

classical mathematician ☹

intuitionistic mathematician ☹

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In this talk

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How about ecumenical typing and verification?

Outline

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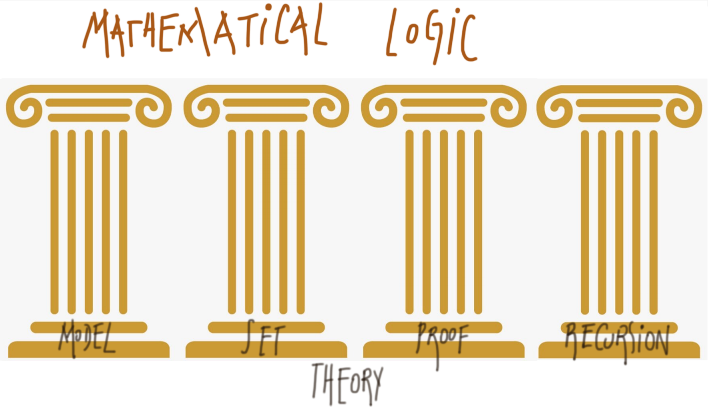
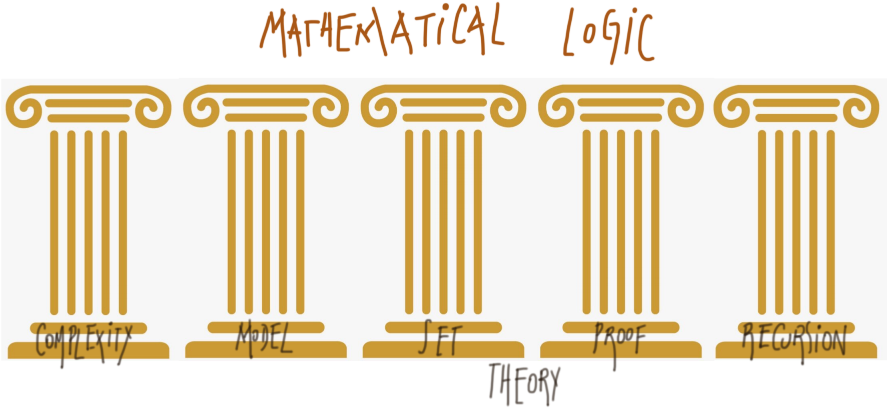
What is Proof Theory?




MATHEMATICAL LOGIC
COMPLEXITY
MODEL
SET
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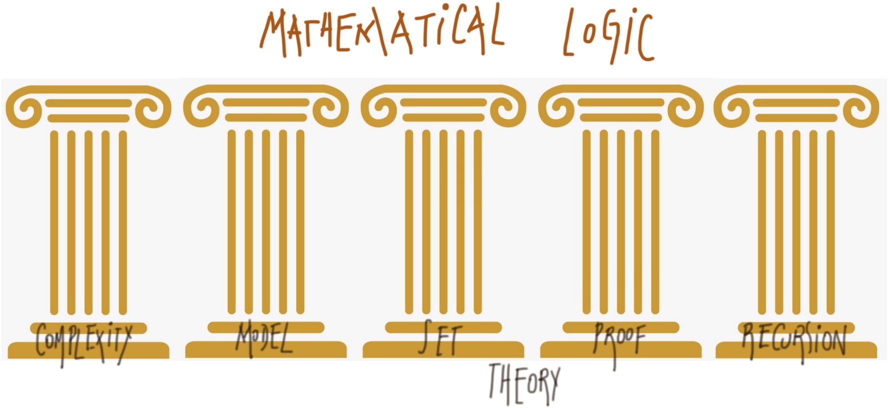
RECURSION
THEORY

What is Proof Theory?



discipline	mathematical objects	(some) tools
set theory	sets	functions
model theory	models & theories	definable & type-definable sets
complexity theory	algorithms	time & memory
recursion theory	computable functions	algorithms
proof theory	proofs	formalisms

Proof theory according to Sonia Marin

It is all about proofs:

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- can we transform one proof into another?
- can we identify patterns?
- for answering all this: formalisation of proofs in a purely mathematical language;
- discipline: proof theory;
- Applications: automatic theorem provers/checkers; extract algorithms from a proof; extract counter-examples from failed proof-search (proof mining); extract proof systems from counter-examples; determine which axioms are required to prove which theorems (reverse mathematics); determine sizes of the proofs (proof complexity).

Reasoning about Brouwer-Heyting-Kolmogorov conditions

H1 A proof of $A \wedge B$ is given by presenting a proof of A and a proof of B.

$$\frac{A \quad B}{A \wedge B} \wedge I$$

H2 A proof of $A \vee B$ is given by presenting either a proof of A or a proof of B.

$$\frac{A}{A \vee B} \vee I_1 \qquad \frac{B}{A \vee B} \vee I_2$$

H3 A proof of $A \rightarrow B$ is a construction which permits us to transform any proof of A into a proof of B.

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Derivation: tree with vertices labelled by formulas.

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A problem: prove analyticity! (called normalization)

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Another problems: harmony, pure systems, etc...

Some *locality*: sequents keep track of open assumptions

where $\Gamma = A_1, \ldots, A_n$ is the context.

Gentzen: sequent calculus

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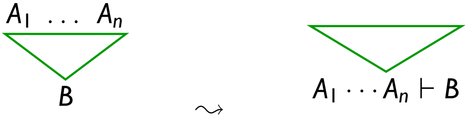

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- Rules: right = introduction rules; left = re-reading elimination rules.

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$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B} \to R \qquad \frac{\Gamma \Rightarrow A \quad \Gamma, B \Rightarrow C}{\Gamma, A \to B \Rightarrow C} \to L$$

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- Analyticity $\rightsquigarrow$ sub-formula property: induces a structure on the proofs (in terms of the end formula).
- Thus, proof structure can be exploited to formalize reasoning, investigate meta-logical properties of the logic e.g. consistency, decidability, complexity and interpolation, and develop automated deduction procedures.

Outline

What is behind Ecumenism?

For a classical logician $A \vee \neg A$ is a theorem.

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But why (and where) do they disagree?

$$\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\ \text{init}}{\Rightarrow A, \neg A}\ \neg R}{\Rightarrow A \vee \neg A}\ \vee R \qquad \dfrac{\dfrac{\overset{?}{A \Rightarrow \bot}}{\Rightarrow \neg A}\ \neg R}{\Rightarrow A \vee \neg A}\ \vee R_2$$

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A solution: They are not talking about the same connective(s) (Prawitz 2015)

> *"The classical logician is not asserting what the intuitionistic logician denies: The classical logician asserts*
>
> $$A \vee_c \neg A$$
>
> *to which the intuitionist does not object; He objects to the universal validity of*
>
> $$A \vee_i \neg A,$$
>
> *which is not asserted by the classical logician."*

Prawitz's ecumenical natural deduction system

Classical

$$\dfrac{\begin{array}{c}[A, \neg B] \\ \Pi \\ \bot\end{array}}{A \rightarrow_c B} \rightarrow_c\text{-int}$$

$$\dfrac{\begin{array}{c}[\neg A, \neg B] \\ \Pi \\ \bot\end{array}}{A \vee_c B} \vee_c\text{-int}$$

$$\dfrac{\begin{array}{c}[\forall x.\neg A] \\ \Pi \\ \bot\end{array}}{\exists_c x.A} \exists_c\text{-int}$$

Shared

$$\dfrac{\begin{array}{c}[A] \\ \Pi \\ \bot\end{array}}{\neg A} \neg\text{-int}$$

$$\dfrac{A \quad B}{A \wedge B} \wedge\text{-int}$$

$$\dfrac{A(a/x)}{\forall x.A} \forall\text{-int}$$

Intuitionistic

$$\dfrac{\begin{array}{c}[A] \\ \Pi \\ B\end{array}}{A \rightarrow_i B} \rightarrow_i\text{-int}$$

$$\dfrac{A_j}{A_1 \vee_i A_2} \vee_i^j\text{-int}$$

$$\dfrac{A(a/x)}{\exists_i x.A} \exists_i\text{-int}$$

(Prawitz 2015)

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$$\dfrac{A(a/x)}{\exists_i x.A} \; \exists_i\text{-int}$$

(Prawitz 2015)

Our ecumenical sequent system LE

$$\frac{\Gamma, A, \neg B \Rightarrow \bot}{\Gamma \Rightarrow A \rightarrow_c B} \rightarrow_c R \qquad \frac{\Gamma, A \Rightarrow \bot}{\Gamma \Rightarrow \neg A} \neg R \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow_i B} \rightarrow_i R$$

$$\frac{\Gamma, \neg A, \neg B \Rightarrow \bot}{\Gamma \Rightarrow A \vee_c B} \vee_c R \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \wedge R \qquad \frac{\Gamma \Rightarrow A_j}{\Gamma \Rightarrow A_1 \vee_i A_2} \vee_i R_j$$

$$\frac{\Gamma, \forall x.\neg A \Rightarrow \bot}{\Gamma \Rightarrow \exists_c x.A} \exists_c R \qquad \frac{\Gamma \Rightarrow A[y/x]}{\Gamma \Rightarrow \forall x.A} \forall R \qquad \frac{\Gamma \Rightarrow A[a/x]}{\Gamma \Rightarrow \exists_i x.A} \exists_i R$$

Classical	**Shared**	Intuitionistic

(Pimentel, Pereira, de Paiva 2021)

Our ecumenical sequent system LE

$$\frac{\Gamma, A, \neg B \Rightarrow \bot}{\Gamma \Rightarrow A \to_c B} \to_c R \qquad \frac{\Gamma, A \Rightarrow \bot}{\Gamma \Rightarrow \neg A} \neg R \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to_i B} \to_i R$$

$$\frac{\Gamma, \neg A, \neg B \Rightarrow \bot}{\Gamma \Rightarrow A \vee_c B} \vee_c R \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \wedge R \qquad \frac{\Gamma \Rightarrow A_j}{\Gamma \Rightarrow A_1 \vee_i A_2} \vee_i R_j$$

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Classical **Shared** **Intuitionistic**

(Pimentel, Pereira, de Paiva 2021)

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$$\frac{\Gamma, A, \neg B \Rightarrow \bot}{\Gamma \Rightarrow A \to_c B} \to_c R \qquad \frac{\Gamma, A \Rightarrow \bot}{\Gamma \Rightarrow \neg A} \neg R \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to_i B} \to_i R$$

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Classical **Shared** **Intuitionistic**

(Pimentel, Pereira, de Paiva 2021)

Back to our mathematical motivation

$$
\dfrac{
  \dfrac{
    \dfrac{
      \overline{\overline{x < z, y < z \Rightarrow x + y \neq 2z}}
      \qquad
      \overline{\overline{x + y = 2z, x + y \neq 2z \Rightarrow \bot}}
    }{
      x + y = 2z, x < z, y < z \Rightarrow \bot
    }\ \text{cut}
  }{
    x + y = 2z \Rightarrow x \geq z \vee_c y \geq z
  }\ \vee_c R
}{
  \Rightarrow x + y = 2z \rightarrow_i x \geq z \vee_c y \geq z
}\ \rightarrow_i R
$$

Theorem

$\Gamma \Rightarrow A$ is provable in LE iff $\vdash_{\mathsf{LE}} \bigwedge \Gamma \rightarrow_i A$.

- The ecumenical entailment is intuitionistic!
- That is, even though some formulas carry with them the notion of classical truth, the logical consequence is intrinsically intuitionist.

Theorem

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- That is, even though some formulas carry with them the notion of classical truth, the logical consequence is intrinsically intuitionist.
- As it should be, since the ecumenical system embeds the classical behavior into intuitionistic logic. ☺
- But if A is classical, the entailment can be read classically.
- And this justifies, proof-theoretically, the ecumenical view of entailments in Prawitz's original proposal.

Outline

The definition of classical connectives depend on other connectives:

$$\cfrac{\begin{matrix}[\forall x.\neg A] \\ \Pi \\ \bot\end{matrix}}{\exists_c x.A}\ \exists_c\text{-int} \qquad \frac{\Gamma, \forall x.\neg A \Rightarrow \bot}{\Gamma \Rightarrow \exists_c x.A}\ \exists_c R$$

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Purifying systems:

- Polarities

$$\dfrac{\Gamma_1 \Rightarrow \Delta_1, P \quad \Gamma_2 \Rightarrow \Delta_2, Q}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, P \wedge Q}\ \wedge_P \qquad \dfrac{\Gamma \Rightarrow \Delta, N \quad \Gamma \Rightarrow \Delta, M}{\Gamma \Rightarrow \Delta, M \wedge N}\ \wedge_N$$

- Stoup

$$\dfrac{\Gamma \Rightarrow \Delta; P}{\Gamma \Rightarrow \Delta, P; \cdot}\ \text{D} \qquad \dfrac{\Gamma \Rightarrow \Delta, N; \cdot}{\Gamma \Rightarrow \Delta; N}\ \text{store}$$

$$\dfrac{\begin{array}{c}[\cdot; A]\\ \Pi\\ \Delta, B; \cdot\end{array}}{\Delta; A \to_c B}\ \to_c\text{-int}$$

$$\dfrac{\Delta, A, B; \cdot}{\Delta; A \vee_c B}\ \vee_c\text{-int}$$

$$\dfrac{\Delta, \exists_c x.A; A(a/x)}{\Delta; \exists_c x.A}\ \exists_c\text{-int}$$

Classical

$$\dfrac{\begin{array}{c}[\cdot; A]\\ \Pi\\ \Delta; \cdot\end{array}}{\Delta; \neg A}\ \neg\text{-int}$$

$$\dfrac{\Delta_1; A \quad \Delta_2; B}{\Delta_1, \Delta_2; A \wedge B}\ \wedge\text{-int}$$

$$\dfrac{\Delta; A(a/x)}{\Delta; \forall x.A}\ \forall\text{-int}$$

Shared

$$\dfrac{\begin{array}{c}[\cdot; A]\\ \Pi\\ \Delta; B\end{array}}{\Delta; A \to_i B}\ \to_i\text{-int}$$

$$\dfrac{\Delta; A_j}{\Delta; A_1 \vee_i A_2}\ \vee_i^j\text{-int}$$

$$\dfrac{\Delta; A(a/x)}{\Delta; \exists_i x.A}\ \exists_i\text{-int}$$

Intuitionistic

(Pereira & Pimentel 2023)

The idea:

$$\vdash_{NE} \Gamma \Rightarrow \Delta; \Sigma \quad \text{iff} \quad \vdash_{LE} \Gamma, \neg\Delta \Rightarrow \Sigma$$

Typical proof:

$$
\begin{prooftree}
\AxiomC{$[\neg A]^2$}
\AxiomC{$[A]^3$}
\RightLabel{$\neg$-elim}
\BinaryInfC{$\bot$}
\RightLabel{DN}
\UnaryInfC{B}
\LeftLabel{3}
\RightLabel{$\rightarrow$-int}
\UnaryInfC{$A \rightarrow B$}
\AxiomC{$[(A \rightarrow B) \rightarrow A]^1$}
\RightLabel{$\rightarrow$-elim}
\BinaryInfC{A}
\AxiomC{$[\neg A]^2$}
\RightLabel{$\neg$-elim}
\BinaryInfC{$\bot$}
\LeftLabel{2}
\RightLabel{DN}
\UnaryInfC{A}
\LeftLabel{1}
\RightLabel{$\rightarrow$-int}
\UnaryInfC{$((A \rightarrow B) \rightarrow A) \rightarrow A$}
\end{prooftree}
$$

Ecumenical stoup:

$$
3\;\dfrac{2\;\dfrac{[\cdot; (A \to_c B) \to_c A]^3 \qquad 1\;\dfrac{\dfrac{[\cdot; A]^1}{A, B; \cdot}\,\text{der}}{A; A \to_c B}\,\to_c\text{-int} \qquad \dfrac{[\cdot; A]^2}{A; \cdot}\,\text{der}}{A; \cdot}\,\to_c\text{-elim}}{\cdot; ((A \to_c B) \to_c A) \to_c A}\,\to_c\text{-int}
$$

Ecumenical stoup:

$$
{}^{3}\,\dfrac{{}^{2}\,\dfrac{[\cdot; (A \rightarrow_c B) \rightarrow_c A]^3 \qquad {}^{1}\,\dfrac{\dfrac{[\cdot; A]^1}{A, B; \cdot}\,\text{der}}{A; A \rightarrow_c B}\,\rightarrow_c\text{-int} \qquad \dfrac{[\cdot; A]^2}{A; \cdot}\,\text{der}}{A; \cdot}\,\rightarrow_c\text{-elim}}{\cdot; ((A \rightarrow_c B) \rightarrow_c A) \rightarrow_c A}\,\rightarrow_c\text{-int}
$$

More interestingly:

$$\vdash_{\mathcal{LE}} \cdot; ((A \rightarrow_j B) \rightarrow_k A) \rightarrow_c A$$

with $j, k \in \{i, c\}$.

Application I: term calculus

The design of the proof system is not only a matter of taste: adequate proposals for extensions and/or applications.

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Michel Parigot (trying to establish a link between control operators and classical constructs):

> *"The difficulties met in trying to use $\neg\neg A \to A$ (or the classical absurdity rule) as a type for control operators is not really due to classical logic, but much more to the deduction system in which it is expressed. It is not easy to find a satisfactory notion of reduction in usual natural deduction because of the restriction to one conclusion which forbids the most natural transformations of proofs."*

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Parigot's solution: adopt a system with stoup!

PARIGOT'S $\lambda\mu$

$$\frac{u : A^{x}, \Gamma \vdash B, \Sigma}{\lambda x.u : \Gamma \vdash A \to B, \Sigma}
\qquad
\frac{t : \Gamma \vdash A \to B, \Sigma \quad u : \Gamma' \vdash A, \Sigma'}{(t\,u) : \Gamma, \Gamma' \vdash B, \Sigma, \Sigma'}$$

$$\frac{t : \Gamma \vdash A, \Sigma}{[\alpha]\,t : \Gamma \vdash A^{\alpha}, \Sigma}
\qquad
\frac{e : \Gamma \vdash A^{\alpha}, \Sigma}{\mu\alpha.e : \Gamma \vdash A, \Sigma}$$

This is dereliction

this adds any formula to the classical context

This is the sole responsible for $\lambda\mu$ being classical

WE CANNOT HAVE THIS RULE!

Ecumenical term calculus

$$\dfrac{}{\Gamma, x : A \vdash x : A \,;\, \Delta}\ \texttt{ax}$$

$$\dfrac{\Gamma, x : A \vdash t : B \,;\, \Delta}{\Gamma \vdash \lambda x.\, t : A \to_i B \,;\, \Delta}\ \texttt{I-}{\to_i} \qquad \dfrac{\Gamma \vdash t : A \to_i B \,;\, \Delta \quad \Gamma \vdash s : A \,;\, \Delta \quad \Gamma, x : B \vdash r : C \,;\, \Delta}{\Gamma \vdash t\,(s, x.r) : C \,;\, \Delta}\ \texttt{E-}{\to_i}$$

$$\dfrac{\Gamma, x : A \vdash c : \bot \,;\, \Delta \cup \{\alpha : B\}}{\Gamma \vdash \mu(x, \alpha).\, c : A \to_c B \,;\, \Delta}\ \texttt{I-}{\to_c} \qquad \dfrac{\Gamma \vdash t : A \to_c B \,;\, \Delta \quad \Gamma \vdash s : A \,;\, \Delta \quad \Gamma, x : B \vdash c : \bot \,;\, \Delta}{\Gamma \vdash t\,[s, x.c] : \bot \,;\, \Delta}\ \texttt{E-}{\to_c}$$

$$\dfrac{\Gamma \vdash t : A \,;\, \Delta}{\Gamma \vdash [\alpha]\, t : \bot \,;\, \Delta \cup \{\alpha : A\}}\ \texttt{der} \qquad \dfrac{\Gamma \vdash c : \bot \,;\, \Delta}{\Gamma \vdash \# c : B \,;\, \Delta}\ \texttt{W}_i$$

Application II: verification

Rewriting Logic (Maude)

```
rl [tensorR] : Gamma, Delta |- F x G => (Gamma |- F) , (Delta |-G) .
```

Rewriting Logic (Maude)

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Gap between what is represented and its representation

Rewriting Logic can rightfully be said to have "ϵ-representational distance" as a semantic and logical framework. (José Meseguer)

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Rewriting logic: Equational theory + rewriting rules

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Rewriting logic: Equational theory + rewriting rules

L-framework (invertibility):

- Case rule $\vee_i R1$

$$\frac{\mathtt{h}_1 : \Delta_2 \vdash \Delta_3 @ \mathtt{F}_4}{\bullet \mathtt{h}_1 : \Delta_2 \vdash \Delta_3 @ \mathtt{F}_4 \vee_i \mathtt{F}_5} \vee_i R1 \quad \rightsquigarrow \quad \frac{\dfrac{}{\mathtt{h}_1 : \Delta_2 \vdash \Delta_3 @ \mathtt{F}_4}\,\mathtt{ax}}{\bullet \mathtt{h}_1 : \Delta_2 \vdash \Delta_3 @ \mathtt{F}_4}\, height$$

- Case rule $\vee_i R2$

$$\frac{\mathtt{h}_1 : \Delta_2 \vdash \Delta_3 @ \mathtt{F}_5}{\bullet \mathtt{h}_1 : \Delta_2 \vdash \Delta_3 @ \mathtt{F}_4 \vee_i \mathtt{F}_5} \vee_i R2 \quad \rightsquigarrow \quad \frac{}{\bullet \mathtt{h}_1 : \Delta_2 \vdash \Delta_3 @ \mathtt{F}_4}\,\mathtt{fail}$$

Outline

What is Modal Logic?

Carlos ______________________________ handsome.

What is Modal Logic?

Classical logic: truth

Carlos ______is______ handsome.

What is Modal Logic?

Classical logic: truth

Carlos ____is not____ handsome.

Modal logic: qualifies truth

Carlos ___is necessarily___ handsome.

Modal logic: qualifies truth

Carlos ______is necessarily______ handsome.
possibly

What is Modal Logic?

Modal logic: qualifies truth

Carlos ___is necessarily___ handsome.

possibly

alethic interpretation

Modal logic: qualifies truth

Carlos ___is known to be___ handsome.

Modal logic: qualifies truth

Carlos ___is known to be___ handsome. (by me)

↑ epistemic interpretation

Modal logic: qualifies truth

Carlos ___*is believed to be*___ handsome. *(by me)*

↑

doxastic interpretation

Modal logic: qualifies truth

Carlos __is obliged to be__ handsome.

Modal logic: qualifies truth

Carlos ___is obliged to be___ handsome.

permission

prohibition

← deontic interpretation

Modal logic: qualifies truth

Carlos ___*is now*___ handsome.

Modal logic: qualifies truth

Carlos ~~is now~~ __________ handsome.

will be

temporal interpretation

Alethic interpretation

Carlos _is necessarily_ handsome.

Alethic interpretation

necessarily Carlos is ________________ handsome.

Alethic interpretation

p = Carlos is handsome

necessarily p

Modalities and propositions

Alethic interpretation

p = Carlos *is* handsome

$$\Box p$$

Alethic interpretation

Carlos <u>*is possibly*</u> handsome.

Alethic interpretation

possibly Carlos <u>is ____________________</u> handsome.

Alethic interpretation

p = Carlos is handsome

possibly p

Alethic interpretation

p = Carlos *is* handsome

$$\Diamond p$$

Truth table

A	B	$A \to B$
1	1	1
1	0	0
0	1	1
0	0	1

Truth table

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

Truth tables

w

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

Generalizing

w

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

v

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

Adding relations

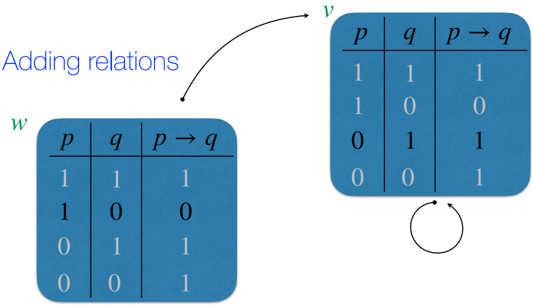
v

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

w

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

Adding relations



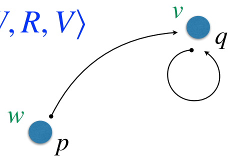
w

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

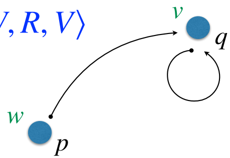
v

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

$\mathscr{M} = \langle W, R, V \rangle$

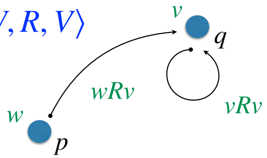


$\mathscr{M} = \langle W, R, V \rangle$

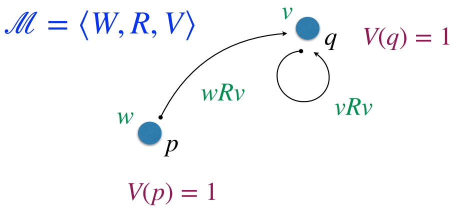


W is a non-empty set of possible worlds.

$\mathscr{M} = \langle W, R, V \rangle$

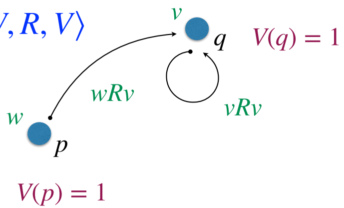


R is the relative accessibility relation:
from the point of view of w, v is possible.



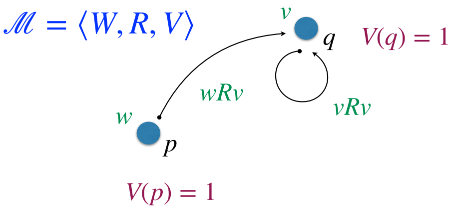
V assigns a truth value to a propositional variable at a world.

$\mathcal{M} = \langle W, R, V \rangle$



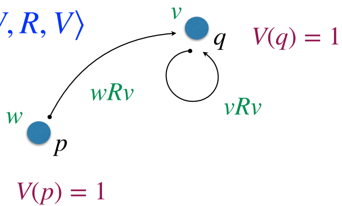
For non-atomic propositional formulas:
Just check the truth table
in each world!

Relational models



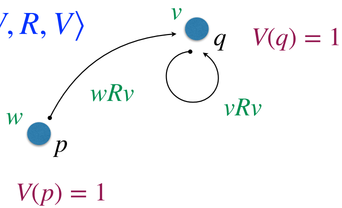
$\mathscr{M}, w \not\models p \to q$ $\mathscr{M}, v \models p \to q$

$\mathscr{M} = \langle W, R, V \rangle$



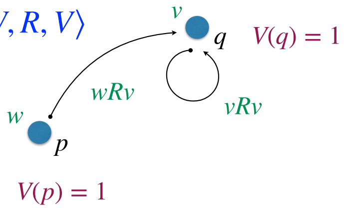
How about modal formulas?

$\mathscr{M} = \langle W, R, V \rangle$



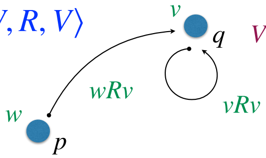
A is *necessary at a world* u provided A is true at every possible world from u.

$\mathscr{M} = \langle W, R, V \rangle$



A is *possible at a world* $\boldsymbol{u}$ provided A is true at some possible world from $\boldsymbol{u}$.

$\mathscr{M} = \langle W, R, V \rangle$



$V(q) = 1$

$V(p) = 1$

$\mathscr{M}, w \nvDash \Box p$ $\qquad$ $\mathscr{M}, v \nvDash \Box p$

$\mathscr{M}, w \vDash \Box q$ $\qquad$ $\mathscr{M}, v \vDash \Box q$

$\mathscr{M}, w \vDash \Box (p \to q)$ $\qquad$ $\mathscr{M}, v \vDash \Box (p \to q)$

$\mathcal{M}, w \Vdash p$	iff	$p \in V(w)$;
$\mathcal{M}, w \Vdash \bot$		never holds;
$\mathcal{M}, w \Vdash \neg A$	iff	$\forall v \geq w. \mathcal{M}, v \nVdash A$;
$\mathcal{M}, w \Vdash A \wedge B$	iff	$\mathcal{M}, w \Vdash A$ and $\mathcal{M}, w \Vdash B$;
$\mathcal{M}, w \Vdash A \vee B$	iff	$\mathcal{M}, w \Vdash A$ or $\mathcal{M}, w \Vdash B$;
$\mathcal{M}, w \Vdash A \rightarrow B$	iff	$\mathcal{M}, w \nVdash A$ or $\mathcal{M}, w \Vdash B$;
$\mathcal{M}, w \Vdash \Box A$	iff	for all v. wRv implies $\mathcal{M}, v \Vdash A$;
$\mathcal{M}, w \Vdash \Diamond A$	iff	there exists v. wRv and $\mathcal{M}, v \Vdash A$.

Ecumenical modalities

$$[\Box A]_x = \forall y (R(x,y) \to [A]_y) \qquad [\Diamond A]_x = \exists y (R(x,y) \wedge [A]_y)$$

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$$\begin{array}{lll} \mathcal{M}, w \models \Box A & \text{iff} & \text{for all } v \text{ such that } wRv, \mathcal{M}, v \models A. \\ \mathcal{M}, w \models \Diamond A & \text{iff} & \text{there exists } v \text{ such that } wRv \text{ and } \mathcal{M}, v \models A. \end{array}$$

$R(x, y)$ represents the accessibility relation R in a Kripke frame.

Ecumenical modalities

$$[\Box A]_x = \forall y(R(x,y) \to [A]_y) \qquad [\Diamond A]_x = \exists y(R(x,y) \wedge [A]_y)$$

$$\vdash_{OL} A \quad \text{iff} \quad \vdash_{ML} \forall x.[A]_x$$

- ML = classical logic $\rightsquigarrow$ OL = classical modal logic K.
- ML = intuitionistic logic $\rightsquigarrow$ OL = intuitionistic modal logic IK.
- ML = Ecumenical logic $\rightsquigarrow$ OL = Ecumenical modal logic EK.

Ecumenical modalities

$$[\Box A]^e_x = \forall y (R(x,y) \to_i [A]^e_y)$$

$$[\Diamond_i A]^e_x = \exists_i y (R(x,y) \wedge [A]^e_y) \qquad [\Diamond_c A]^e_x = \exists_c y (R(x,y) \wedge [A]^e_y)$$

Ecumenical modalities

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$$[\Diamond_i A]^e_x = \exists_i y(R(x,y) \wedge [A]^e_y) \qquad [\Diamond_c A]^e_x = \exists_c y(R(x,y) \wedge [A]^e_y)$$

- $\Diamond_c A \leftrightarrow_i \neg\Box\neg A$ but $\Diamond_i A \not\leftrightarrow_i \neg\Box\neg A$.
- Restricted to the classical fragment: $\Box$ and $\Diamond_c$ are duals.

Ecumenical Modal Logic

- Formulas: $A ::= p_i \mid p_c \mid \bot \mid A \wedge A \mid A \vee_i A \mid A \vee_c A \mid A \rightarrow_i A \mid A \rightarrow_c A \mid \Box A \mid \Diamond_i A \mid \Diamond_c A$
- Independence of the modalities
- Axioms: ecumenical propositional logic and

$$
\begin{array}{l}
\mathsf{k}_1 : \Box(A \rightarrow_i B) \rightarrow_i (\Box A \rightarrow_i \Box B) \\
\mathsf{k}_2 : \Box(A \rightarrow_i B) \rightarrow_i (\Diamond_i A \rightarrow_i \Diamond_i B) \\
\mathsf{k}_3 : \Diamond_i(A \vee_i B) \rightarrow_i (\Diamond A \vee_i \Diamond B) \\
\mathsf{k}_4 : (\Diamond_i A \rightarrow_i \Box B) \rightarrow_i \Box(A \rightarrow_i B) \\
\mathsf{k}_5 : \neg \Diamond_i \bot
\end{array}
$$

EK (Marin et al. 2020)

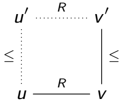
- Rules: modus ponens: $\dfrac{A \quad A \rightarrow B}{B}$ necessitation: $\dfrac{A}{\Box A}$
- Semantics: Ecumenical Birelational structures $(W, R, \leq)$

  a non-empty set W of worlds;

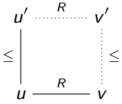
  a binary relation $R \subseteq W \times W$;

  a preorder $\leq$ on W.

(F_1)



(F_2)



Ecumenical Modal Logic

- Formulas: $A ::= p_i \mid p_c \mid \bot \mid A \wedge A \mid A \vee_i A \mid A \vee_c A \mid A \rightarrow_i A \mid A \rightarrow_c A \mid \Box A \mid \Diamond_i A \mid \Diamond_c A$
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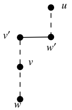

$$\mathcal{M}, w \models_{\mathsf{E}} \Diamond_c A \text{ iff } \forall v \geq w. \exists u. v\,(\leq \circ R \circ \leq)\, u,\ \mathcal{M}, u \models_{\mathsf{E}} A$$

Labeled modal rules:

$$\frac{x : \Box\neg A, \Gamma \vdash x : \bot}{\Gamma \vdash x : \Diamond_c A} \Diamond_c R \qquad \frac{xRy, \Gamma \vdash y : A}{\Gamma \vdash x : \Box A} \Box R \qquad \frac{xRy, \Gamma \vdash y : A}{xRy, \Gamma \vdash x : \Diamond_i A} \Diamond_i R$$

Ecumenical modal proof theory

Labeled modal rules:

$$\frac{x : \Box\neg A, \Gamma \vdash x : \bot}{\Gamma \vdash x : \Diamond_c A}\ \Diamond_c R \qquad \frac{xRy, \Gamma \vdash y : A}{\Gamma \vdash x : \Box A}\ \Box R \qquad \frac{xRy, \Gamma \vdash y : A}{xRy, \Gamma \vdash x : \Diamond_i A}\ \Diamond_i R$$

Labeled modal rules:

$$\frac{x : \Box\neg A, \Gamma \vdash x : \bot}{\Gamma \vdash x : \Diamond_c A} \ \Diamond_c R \qquad \frac{xRy, \Gamma \vdash y : A}{\Gamma \vdash x : \Box A} \ \Box R \qquad \frac{xRy, \Gamma \vdash y : A}{xRy, \Gamma \vdash x : \Diamond_i A} \ \Diamond_i R$$

Ecumenical modal proof theory

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Extensions:

Axiom	Condition	First-Order Formula
T : $\Box A \to_i A \wedge A \to_i \Diamond_i A$	Reflexivity	$\forall x.R(x,x)$
4 : $\Box A \to_i \Box\Box A \wedge \Diamond_i\Diamond_i A \to_i \Diamond_i A$	Transitivity	$\forall x,y,z.(R(x,y) \wedge R(y,z)) \to_i R(x,z)$
5 : $\Box A \to_i \Box\Diamond_i A \wedge \Diamond_i\Box A \to_i \Diamond_i A$	Euclideaness	$\forall x,y,z.(R(x,y) \wedge R(x,z)) \to_i R(y,z)$
B : $A \to_i \Box\Diamond_i A \wedge \Diamond_i\Box A \to_i A$	Symmetry	$\forall x,y.R(x,y) \to_i R(y,x)$

Labeled modal rules:

$$\frac{x : \Box\neg A, \Gamma \vdash x : \bot}{\Gamma \vdash x : \Diamond_c A}\ \Diamond_c R \qquad \frac{xRy, \Gamma \vdash y : A}{\Gamma \vdash x : \Box A}\ \Box R \qquad \frac{xRy, \Gamma \vdash y : A}{xRy, \Gamma \vdash x : \Diamond_i A}\ \Diamond_i R$$

Extensions:

Axiom	Condition	First-Order Formula
$\mathsf{T} : \Box A \to_i A \wedge A \to_i \Diamond_i A$	Reflexivity	$\forall x.R(x,x)$
$4 : \Box A \to_i \Box\Box A \wedge \Diamond_i\Diamond_i A \to_i \Diamond_i A$	Transitivity	$\forall x,y,z.(R(x,y) \wedge R(y,z)) \to_i R(x,z)$
$5 : \Box A \to_i \Box\Diamond_i A \wedge \Diamond_i\Box A \to_i \Diamond_i A$	Euclideaness	$\forall x,y,z.(R(x,y) \wedge R(x,z)) \to_i R(y,z)$
$\mathsf{B} : A \to_i \Box\Diamond_i A \wedge \Diamond_i\Box A \to_i A$	Symmetry	$\forall x,y.R(x,y) \to_i R(y,x)$

Rules:

$$\frac{xRx, \Gamma \vdash w : C}{\Gamma \vdash w : C}\ \mathsf{T} \qquad \frac{xRz, \Gamma \vdash w : C}{xRy, yRz, \Gamma \vdash w : C}\ 4$$

$$\frac{yRz, \Gamma \vdash w : C}{xRy, xRz, \Gamma \vdash w : C}\ 5 \qquad \frac{yRx, \Gamma \vdash w : C}{xRy, \Gamma \vdash w : C}\ \mathsf{B}$$

Outline

Getting rid of negation

LE

$$\Gamma, \neg\Delta \vdash C$$

Getting rid of negation

LE	→	LCE
$\Gamma, \neg\Delta \vdash C$		$\Gamma \vdash \Delta; C$

Getting rid of negation

LE $\longrightarrow$ LCE

$$\Gamma, \neg\Delta \vdash C \qquad\qquad \Gamma \vdash \Delta; C$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow_i B} \rightarrow_i R$$

Getting rid of negation

LE $\longrightarrow$ LCE

$$\Gamma, \neg\Delta \vdash C$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow_i B} \rightarrow_i R$$

$$\Gamma \vdash \Delta; C$$

$$\frac{\Gamma, A \vdash \Delta; B}{\Gamma \vdash \Delta; A \rightarrow_i B} \rightarrow_i R$$

LE → LCE

LE	LCE
$\Gamma, \neg\Delta \vdash C$	$\Gamma \vdash \Delta; C$
$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to_i B} \to_i R$	$\dfrac{\Gamma, A \vdash \Delta; B}{\Gamma \vdash \Delta; A \to_i B} \to_i R$
$\dfrac{\Gamma, A, \neg B \vdash \bot}{\Gamma \vdash A \to_c B} \to_c R$	

LE ⟶ LCE

$$\Gamma, \neg\Delta \vdash C \qquad\qquad \Gamma \vdash \Delta; C$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to_i B} \to_i R \qquad\qquad \frac{\Gamma, A \vdash \Delta; B}{\Gamma \vdash \Delta; A \to_i B} \to_i R$$

$$\frac{\Gamma, A, \neg B \vdash \bot}{\Gamma \vdash A \to_c B} \to_c R \qquad\qquad \frac{\Gamma, A \vdash B, \Delta; \cdot}{\Gamma \vdash A \to_c B, \Delta; \cdot} \to_c R$$

Getting rid of negation

LE → LCE

$$\Gamma, \neg\Delta \vdash C$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to_i B} \to_i R$$

$$\frac{\Gamma, A, \neg B \vdash \bot}{\Gamma \vdash A \to_c B} \to_c R$$

$$\Gamma \vdash \Delta; C$$

$$\frac{\Gamma, A \vdash \Delta; B}{\Gamma \vdash \Delta; A \to_i B} \to_i R$$

$$\frac{\Gamma, A \vdash B, \Delta; \cdot}{\Gamma \vdash A \to_c B, \Delta; \cdot} \to_c R$$

labEK

$$\frac{x : \Box\neg A, \Gamma \Rightarrow x : \bot}{\Gamma \Rightarrow x : \Diamond_c A} \Diamond_c R$$

Getting rid of negation

LE ⟶ LCE

$$\Gamma, \neg\Delta \vdash C \qquad \longrightarrow \qquad \Gamma \vdash \Delta; C$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to_i B} \to_i R \qquad \longrightarrow \qquad \frac{\Gamma, A \vdash \Delta; B}{\Gamma \vdash \Delta; A \to_i B} \to_i R$$

$$\frac{\Gamma, A, \neg B \vdash \bot}{\Gamma \vdash A \to_c B} \to_c R \qquad \longrightarrow \qquad \frac{\Gamma, A \vdash B, \Delta; \cdot}{\Gamma \vdash A \to_c B, \Delta; \cdot} \to_c R$$

labEK ⟶ Pure labEK

$$\frac{x : \Box\neg A, \Gamma \Rightarrow x : \bot}{\Gamma \Rightarrow x : \Diamond_c A} \Diamond_c R \qquad \longrightarrow \qquad \frac{xRy, \Gamma \vdash y : A, x : \Diamond_c A, \Delta; \cdot}{xRy, \Gamma \vdash x : \Diamond_c A, \Delta; \cdot} \Diamond_c R$$

A derivation example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overline{xRy, y : A, x : \neg\Box\neg A \vdash x : \Diamond_c A, y : A; \cdot}\ \text{init}}{xRy, y : A, x : \neg\Box\neg A \vdash x : \Diamond_c A; \cdot}\ \Diamond_c R}{xRy, x : \neg\Box\neg A \vdash x : \Diamond_c A; y : \neg A}\ \neg R}{x : \neg\Box\neg A \vdash x : \Diamond_c A; x : \Box\neg A}\ \Box R}{x : \neg\Box\neg A \vdash x : \Diamond_c A; \cdot}\ \neg L
$$

Getting rid of labels

- Polarities:

$$
\begin{array}{lcl}
N & := & p_c \mid \bot \mid A \vee_c A \mid A \rightarrow_c A \mid \Diamond_c A \\
P & := & p_i \mid A \vee_i A \mid \neg A \mid A \rightarrow_i A \mid A \wedge A \mid \Diamond_i A \mid \Box A
\end{array}
$$

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\end{array}
$$

- Harmony:

$$
\frac{\Gamma \vdash \Delta; x : P \quad x : P, \Gamma \vdash \Delta; \Pi}{\Gamma \vdash \Delta; \Pi}\ \mathsf{cut}_i
\qquad
\frac{\Gamma \vdash \Delta, x : N; \Pi^* \quad x : N, \Gamma \vdash \Delta; \Pi}{\Gamma \vdash \Delta; \Pi}\ \mathsf{cut}_c
$$

where Π^* is either empty or some $y : P \in \Delta$

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$$
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N & := & p_c \mid \bot \mid A \vee_c A \mid A \to_c A \mid \Diamond_c A \\
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$$

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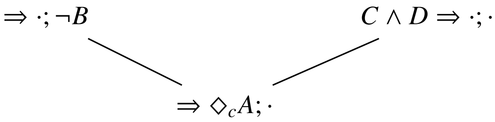
$$
\frac{\Gamma \vdash \Delta; x : P \quad x : P, \Gamma \vdash \Delta; \Pi}{\Gamma \vdash \Delta; \Pi}\ \mathrm{cut}_i
\qquad
\frac{\Gamma \vdash \Delta, x : N; \Pi^* \quad x : N, \Gamma \vdash \Delta; \Pi}{\Gamma \vdash \Delta; \Pi}\ \mathrm{cut}_c
$$

where Π^* is either empty or some $y : P \in \Delta$

- Internal nested systems – no labels! nEK (Marin et al. 2021).

$$xRy, xRz, z : C \wedge D \Rightarrow x : \Diamond_c A; y : \neg B$$

corresponds to the tree of sequents with stoup



Outline

Ecumenical systems may help us to have a better understanding of the relation between classical logic and intuitionistic logics.

♡ Subjects dear to me ♡ – Part I

Ecumenical systems may help us to have a better understanding of the relation between classical logic and intuitionistic logics.

- discuss the precise detection of the parts of a mathematical proof that are intrinsically intuitionistic, classical or independent;
- other approaches: how about negation?
- ecumenical nature of atoms.

Ecumenical systems may help us to have a better understanding of the relation between classical logic and intuitionistic logics.

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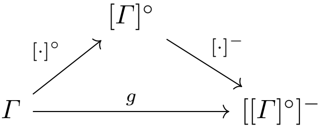
- other approaches: how about negation?
- ecumenical nature of atoms.

- Nothing is said about the basic relations used for generating atomic formulas.
- Should atoms be primitive relations or be defined?
- Moreover: the presence of classical and intuitionistic “interpretations” of predicates entails a double-negation flavor to the system!
- Recent work with Luiz Carlos and Valeria: ecumenical systems with no such interpretations. The constructive interpretation interpolates the Gödel-Gentzen translation:



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- algebraic ecumenical models?

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- other approaches: how about negation?
- ecumenical nature of atoms.
- algebraic ecumenical models?
- Model-theoretic semantics: truth $\times$ Proof-theoretic semantics: proof
- Emphasizes the fundamental nature of proofs.
- Satisfiability of an atomic formula p at a state w in a Kripke model:

$$w \Vdash p \qquad \text{iff} \qquad w \in V(p)$$

Validity w.r.t. a set of atomic rules S in proof-theoretic semantics:

$$\Vdash_S p \qquad \text{iff} \qquad \vdash_S p$$

- Recent work with Victor and Luiz Carlos: proof-theoretic semantics for ecumenical logical systems. Main motto:
Classical proof + monotonicity = intuitionistic proof of double negation.

♡ Subjects dear to me ♡ – Part II

What can we say about modal ecumenical systems?

- constructive modal logic and beyond;
- algebraic ecumenical models?
- ecumenical typing: fragments as well known typed modal systems;
- ecumenical nature of atoms.

What can we say about modal ecumenical systems?

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Connectedness property:

$$(Conn_1) \qquad aRb \vee bRa \qquad (Conn_2) \qquad \neg(a = b) \rightarrow aRb \vee bRa$$

If R is reflexive, then

$$\vdash_{\mathsf{LK}} Conn_2 \rightarrow Conn_1 \text{ but } \nvdash_{\mathsf{LJ}} Conn_2 \rightarrow Conn_1$$

- Background logic = classical logic $\Rightarrow$ the same characterization of $S4.3$ by using $Conn_1$ or $Conn_2$.
- Background logic = intuitionistic logic $\Rightarrow$ two different modal extensions.
- Alberto Naibo: "Would this make any difference at the level of the modal systems that we can characterize using an intuitionistic background logic?"

Thanks!!!

Obrigada!!!

Merci!!!

Gracias!!!

☺