Optics are Monoidal Context

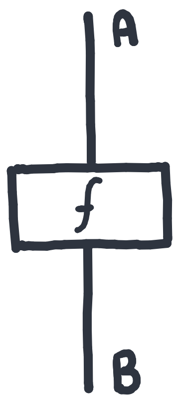
Mario Román joint with Matt Earnshaw and James Hefford

MFPS, 23rd June

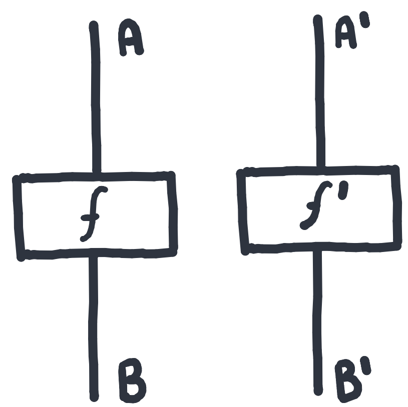
Supported by the EU Estonian IT Academy.

Monoidal Categories: Process Theories

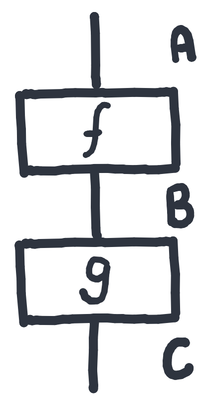
Monoidal categories are an algebra of parallel and sequential composition.
String diagrams are an internal language of monoidal categories.



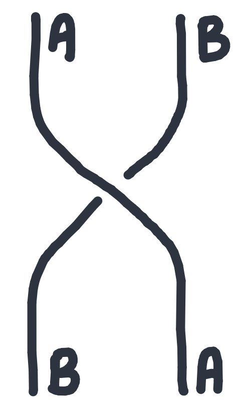
Process



Parallel composition



Sequential composition



Swap

Bènabou

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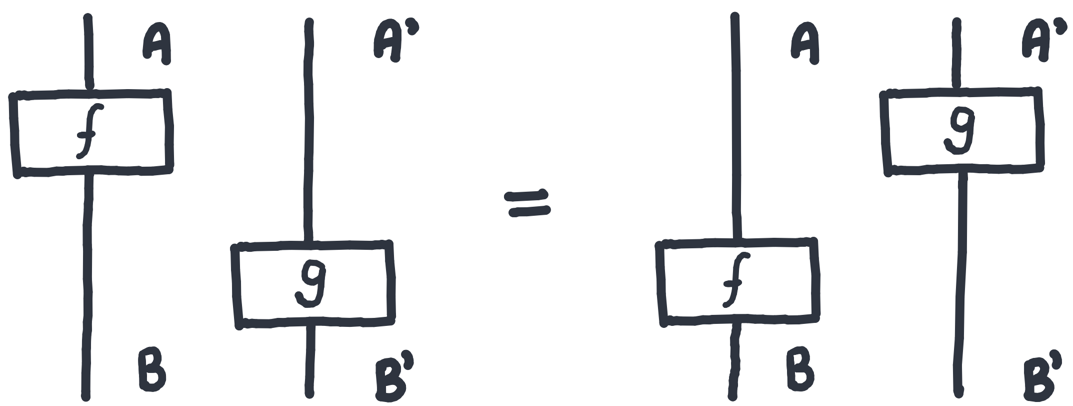


Interchange Law

Bènabou

PART 0: Optics

OPTICS

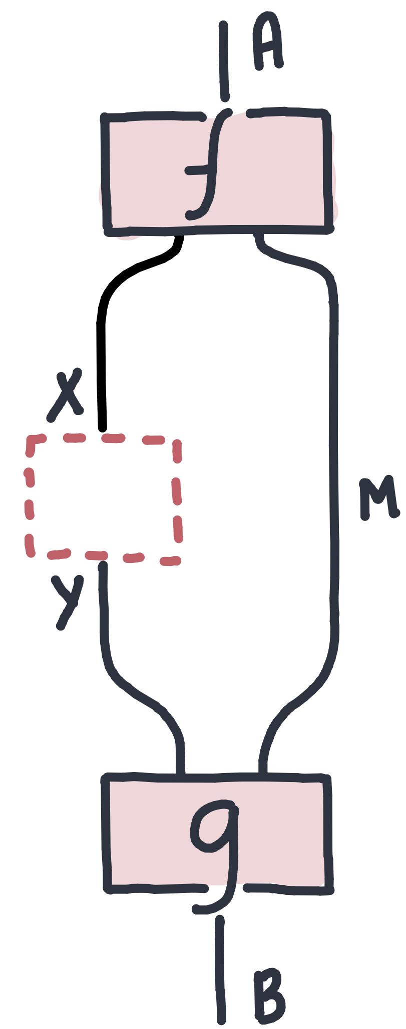
DEFINITION. Let $\mathbb{C}, \otimes, I$ symm. monoidal.

An optic from A to B with a hole from X to Y is a pair of morphisms

$$f : A \to X \otimes M, \qquad g : Y \otimes M \to B,$$

written as $\langle f \mid g \rangle$, and quotiented by dinaturality on M:

$$\langle f ⨾ (id \otimes h) \mid g \rangle = \langle f \mid (id \otimes h) ⨾ g \rangle.$$



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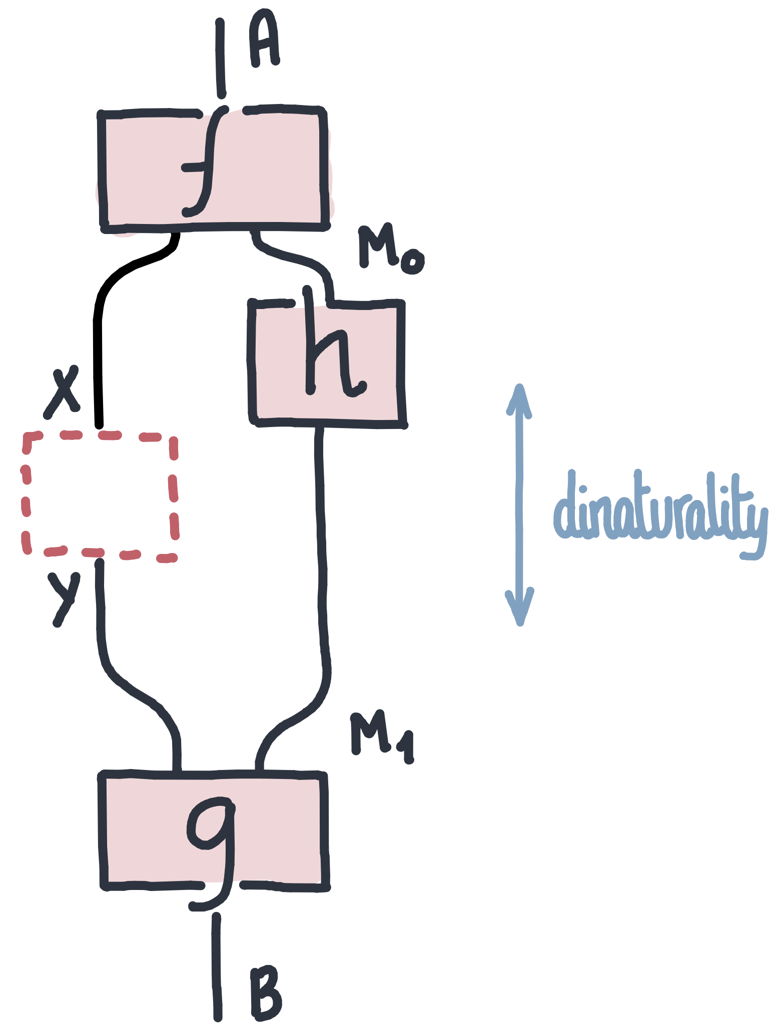

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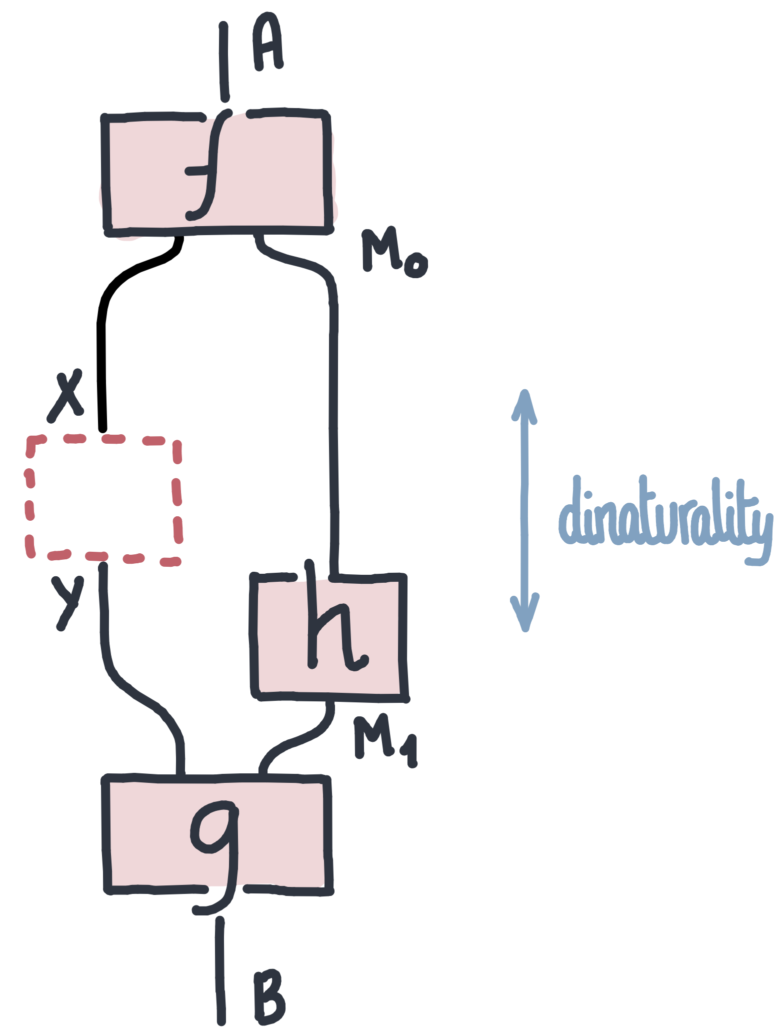
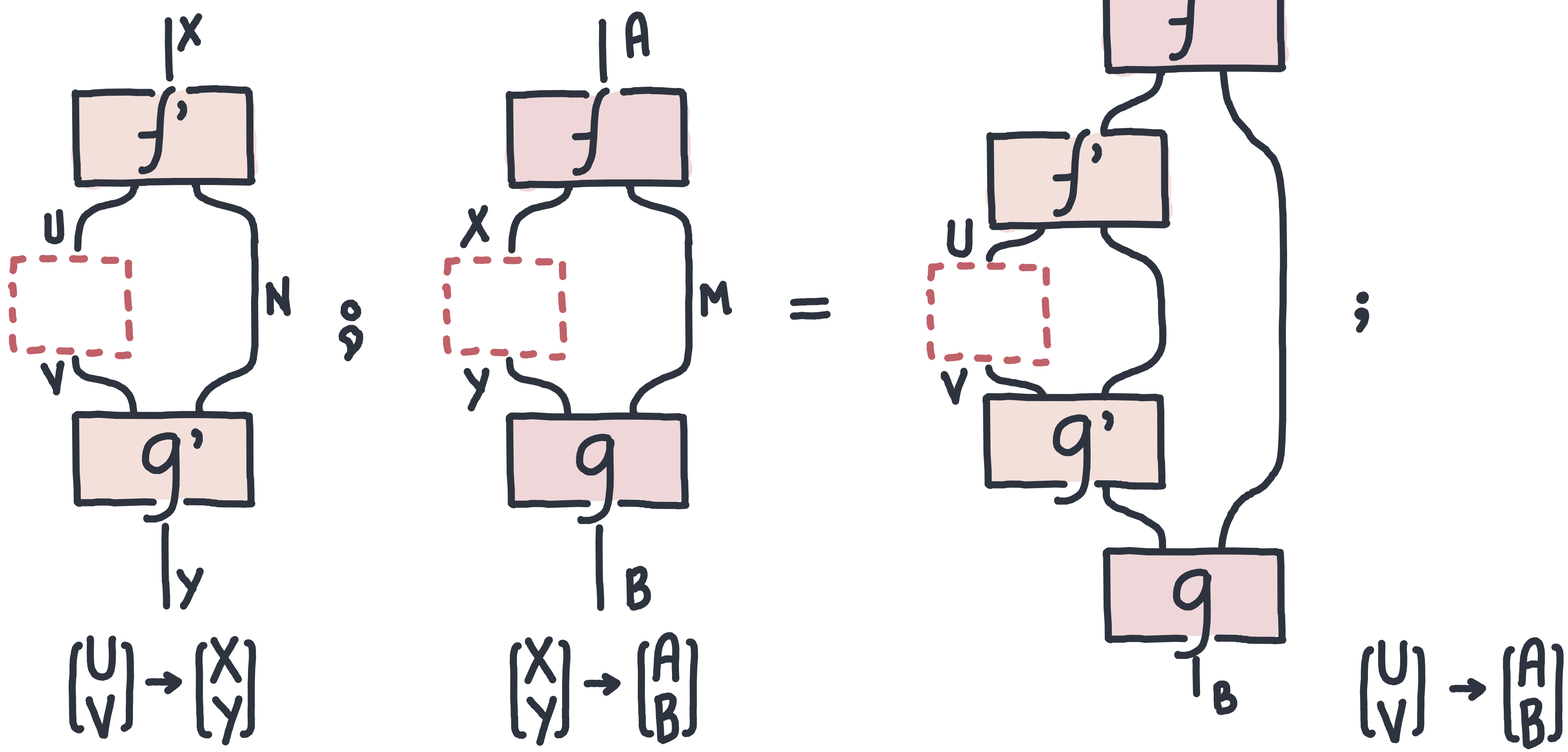


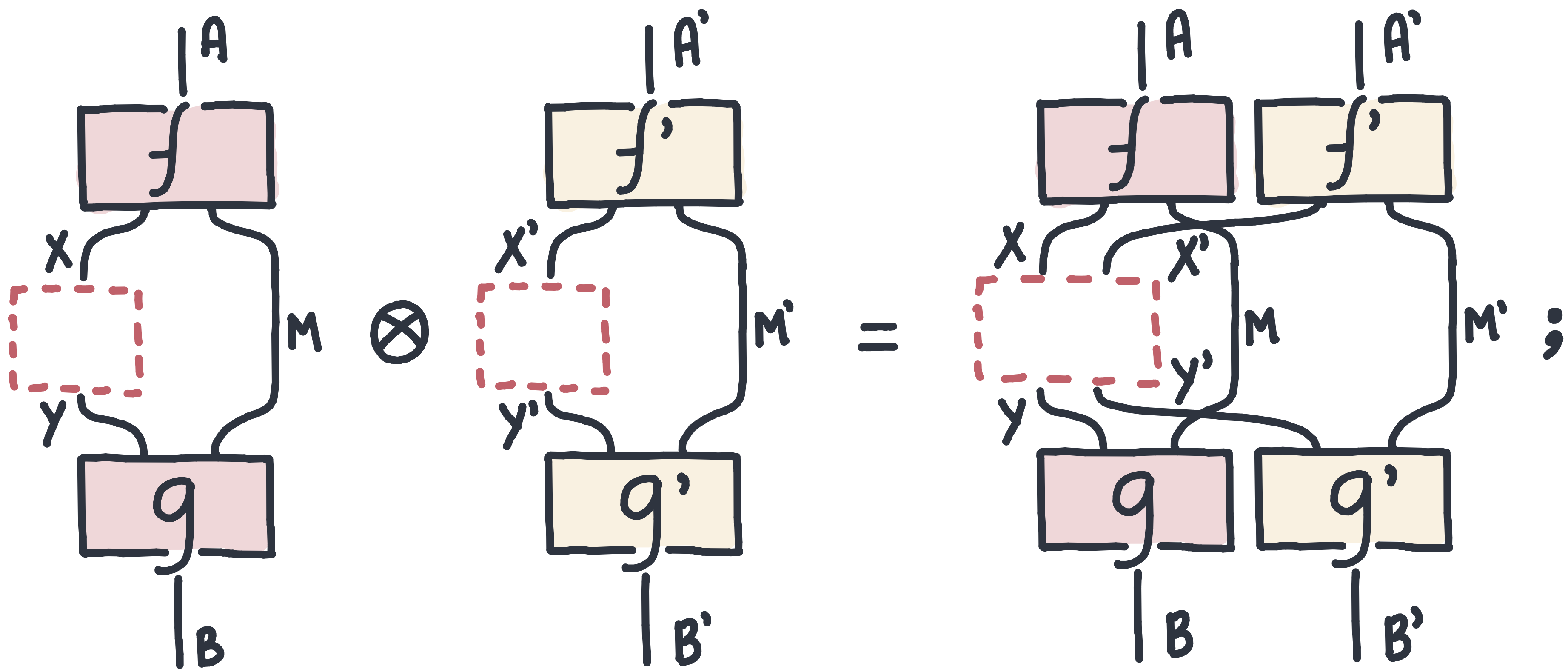
Optics form a Category

Objects are pairs $\begin{bmatrix} X \\ Y \end{bmatrix}$. Composition is



Optics form a Monoidal Category

Tensoring is $\begin{bmatrix} X \\ Y \end{bmatrix} \otimes \begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} X \otimes X' \\ Y \otimes Y' \end{bmatrix}$, and



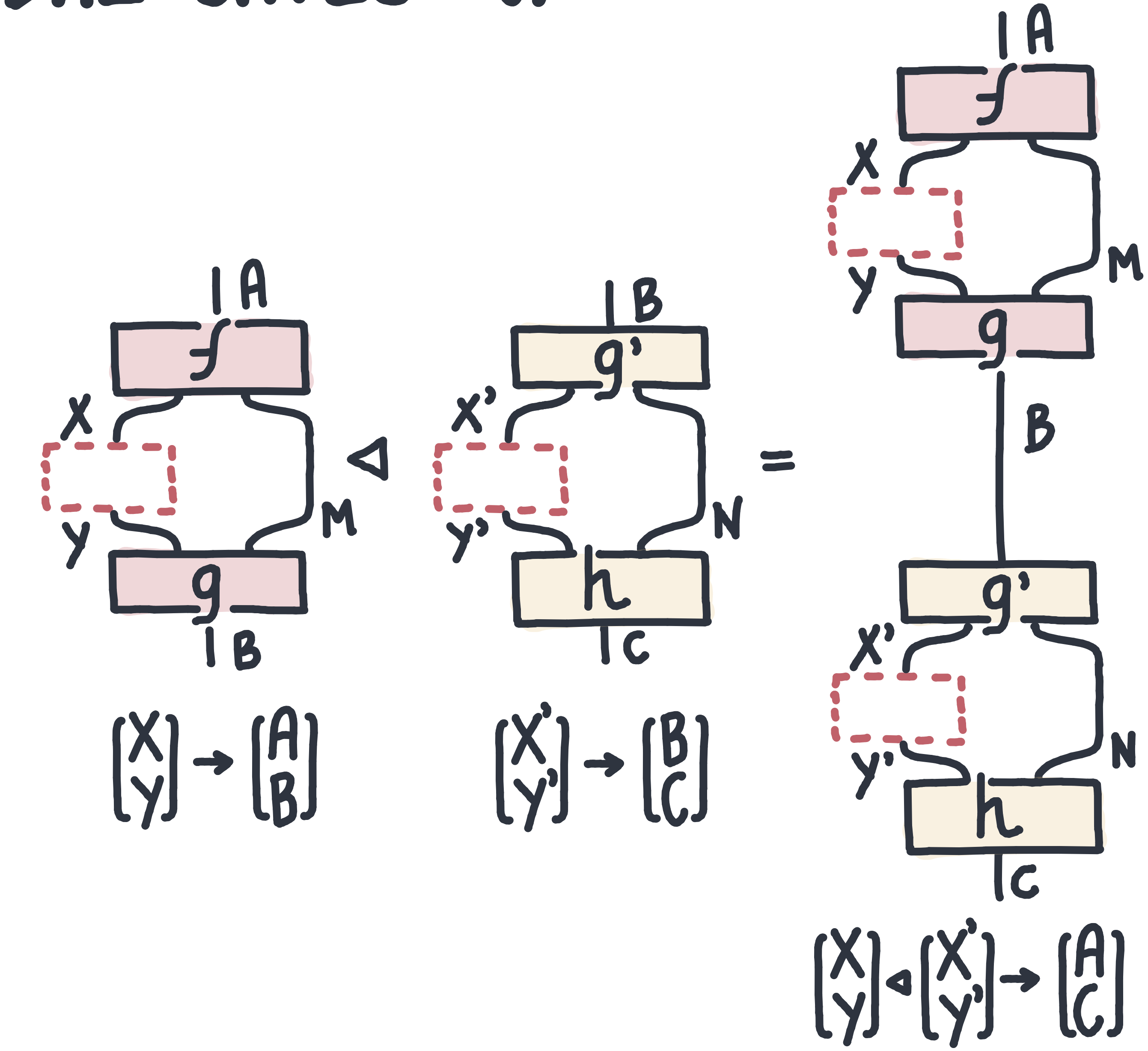
$\begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} A \\ B \end{bmatrix}$ $\quad$ $\begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} A \\ B \end{bmatrix}$ $\quad$ $\begin{bmatrix} X \otimes X' \\ Y \otimes Y' \end{bmatrix} \to \begin{bmatrix} A \otimes A' \\ B \otimes B' \end{bmatrix}$

Optics form a Duoidal Category?

Sequencing is not an operation:

$\begin{bmatrix} X \\ Y \end{bmatrix} \triangleleft \begin{bmatrix} X' \\ Y' \end{bmatrix}$ is not an object, even when $\begin{bmatrix} X \\ Y \end{bmatrix} \triangleleft \begin{bmatrix} X' \\ Y' \end{bmatrix} \to \begin{bmatrix} A \\ B \end{bmatrix}$ is defined.

This is not monoidal, but it is still promonoidal.



Optics form a Duoidal Category?

Sequencing is not an operation, it defines a hom-set to an object that does not really exist.

$\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix}$ is not an object,
but $\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ is defined.

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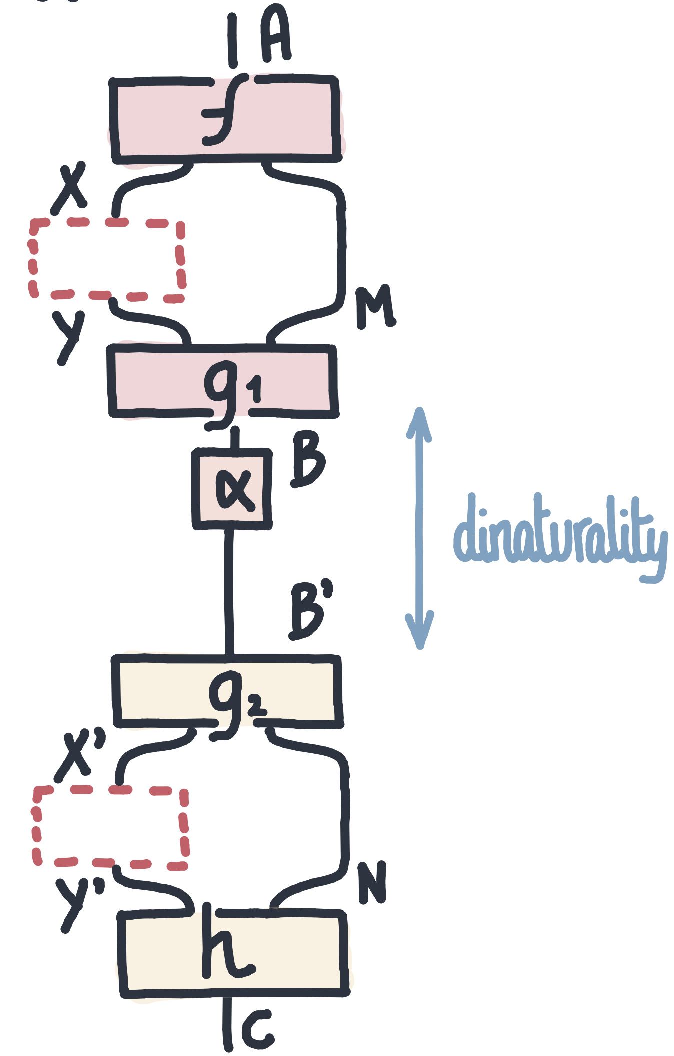

$$\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\C\end{bmatrix}$$

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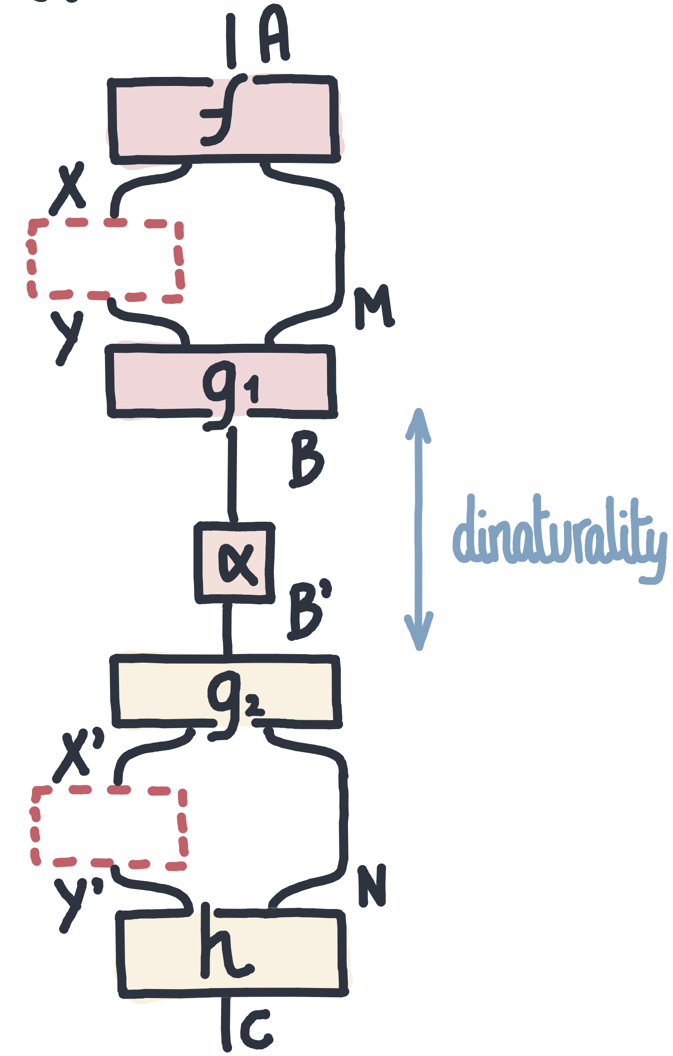

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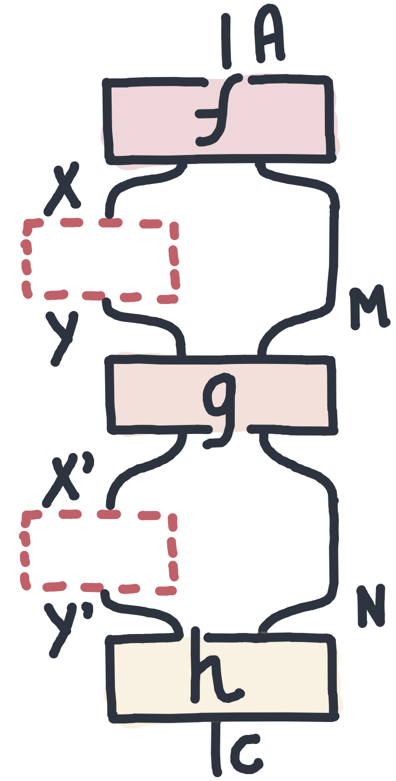

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PART 1 : Promonoidals

Monoidal Category

DEFINITION. A monoidal category is a category $\mathbb{C}$ together with functors

$$(\otimes) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}, \qquad I : \mathbb{1} \to \mathbb{C},$$

and natural isomorphisms

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C,$$
$$\lambda_A : I \otimes A \to A,$$
$$\rho_A : A \otimes I \to A,$$

satisfying the pentagon and triangle equations.

By nesting, $X \otimes (Y \otimes Z)$, we mean functor composition,

$$X \otimes (Y \otimes Z) :=$$
$$X \otimes M \text{ where } M = Y \otimes Z.$$

Promonoidal Category

DEFINITION. A promonoidal category is a category $\mathbb{C}$ together with profunctors

$$\mathbb{C}(\bullet \otimes \bullet ; \bullet) \colon \mathbb{C}^{op} \times \mathbb{C} \times \mathbb{C} \to \mathrm{SET}, \qquad \mathbb{C}(I ; \bullet) \colon \mathbb{C}^{op} \to \mathrm{SET},$$

and natural bijections,

$$\alpha_{A,B,C} \colon \mathbb{C}(X \otimes (Y \otimes Z) ; \bullet) \to \mathbb{C}((X \otimes Y) \otimes Z ; \bullet),$$
$$\lambda_A \colon \mathbb{C}(I \otimes X ; \bullet) \to \mathbb{C}(X ; \bullet),$$
$$\rho_A \colon \mathbb{C}(X \otimes I ; \bullet) \to \mathbb{C}(X ; \bullet),$$

satisfying the pentagon and triangle equations.

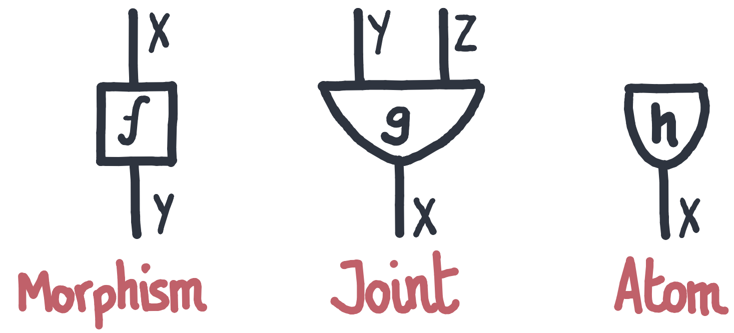
By nesting, $\mathbb{C}(X \otimes (Y \otimes Z) ; \bullet)$, we mean profunctor composition,

$$\mathbb{C}(X \otimes (Y \otimes Z) ; \bullet) := \int^{M} \mathbb{C}(X \otimes M ; \bullet) \times \mathbb{C}(Y \otimes Z ; M).$$

PROMONOIDAL CATEGORIES

Promonoidal categories provide a theory of coherent composition. It has

- Morphisms, $\mathbb{C}(X;A)$.
- Joints, $\mathbb{C}(X \triangleleft Y;A)$.
- Atoms, $\mathbb{C}(N;A)$.



Malleability property: splitting A into X and "something" and then splitting that something into Y and Z can be done in the same number of ways as splitting A into "something" and Z and then splitting that something into X and Y.

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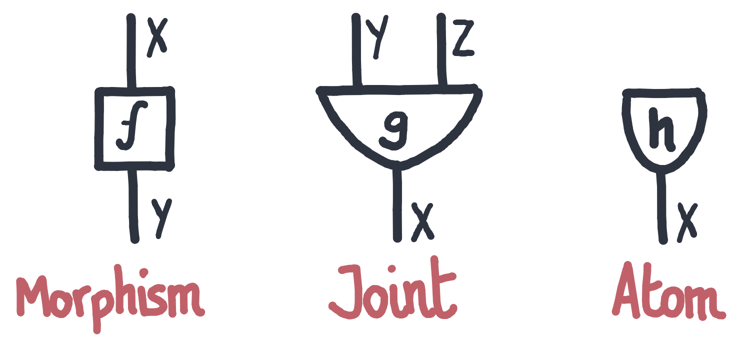
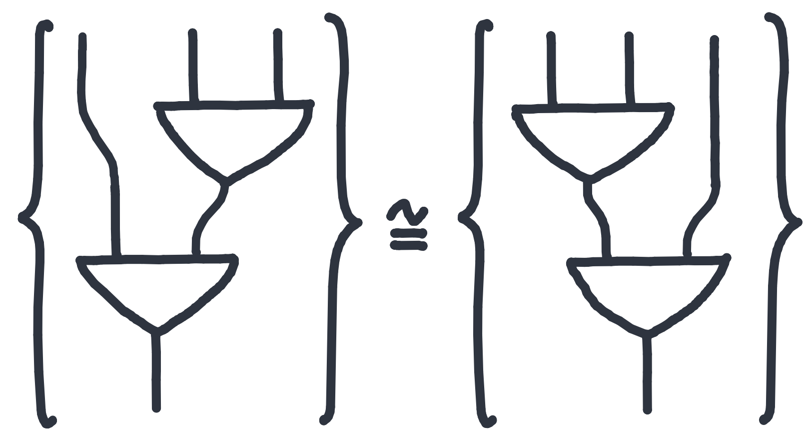


Malleability property:

$$\int^{M\in\mathbb{C}} \mathbb{C}(A;X\otimes M)\times\mathbb{C}(M;Y\otimes Z) \cong \int^{M\in\mathbb{C}} \mathbb{C}(A;M\otimes Z)\times\mathbb{C}(M;X\otimes Y);$$

$$\int^{M\in\mathbb{C}} \mathbb{C}(A;X\otimes M)\times\mathbb{C}(M;I) \cong \mathbb{C}(A;X);$$

$$\int^{M\in\mathbb{C}} \mathbb{C}(A;M\otimes X)\times\mathbb{C}(M;I) \cong \mathbb{C}(A;X);$$



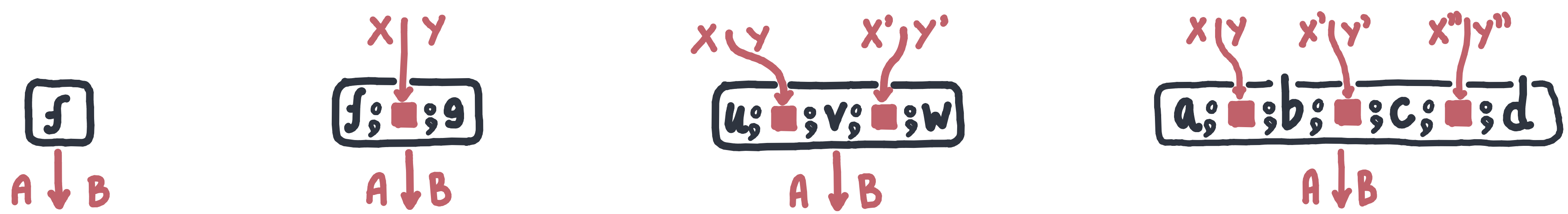
PART 2: Context for Categories

CONTEXT FOR CATEGORIES

Consider 'expressions with holes' in a category, like the following

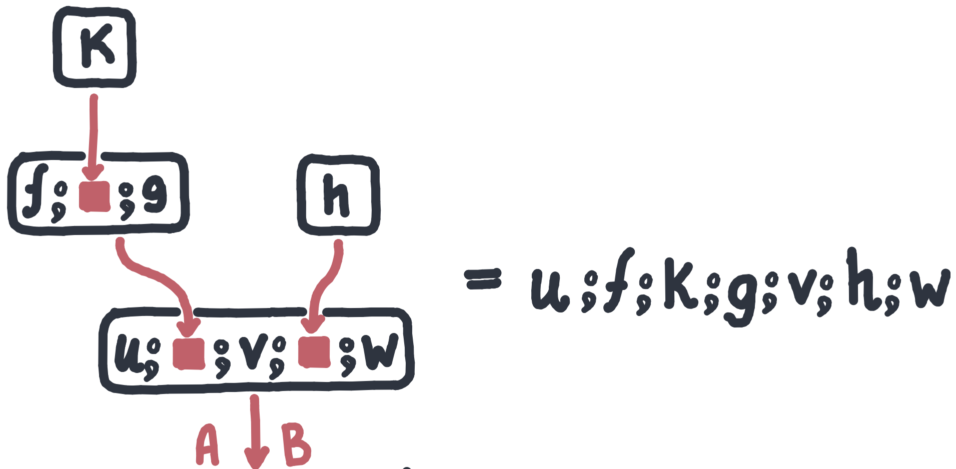
$$u \mathbin{;} \blacksquare \mathbin{;} v \mathbin{;} \blacksquare \mathbin{;} w, \qquad f \mathbin{;} \blacksquare \mathbin{;} g, \qquad f, \qquad a \mathbin{;} \blacksquare \mathbin{;} b \mathbin{;} \blacksquare \mathbin{;} c \mathbin{;} \blacksquare \mathbin{;} d.$$

These contexts form a promonoidal category.



$$\begin{bmatrix}A\\B\end{bmatrix} \qquad \begin{bmatrix}X\\Y\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix} \qquad \begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\C\end{bmatrix} \qquad \begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \triangleleft \begin{bmatrix}X''\\Y''\end{bmatrix} \to \begin{bmatrix}A\\C\end{bmatrix}$$

CONTEXT FOR CATEGORIES



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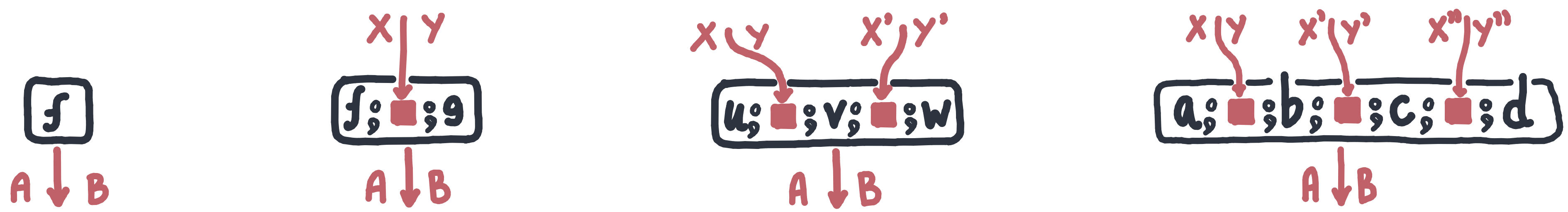


$$\begin{bmatrix} A \\ B \end{bmatrix} \qquad \begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} A \\ B \end{bmatrix} \qquad \begin{bmatrix} X \\ Y \end{bmatrix} \triangleleft \begin{bmatrix} X' \\ Y' \end{bmatrix} \to \begin{bmatrix} A \\ C \end{bmatrix} \qquad \begin{bmatrix} X \\ Y \end{bmatrix} \triangleleft \begin{bmatrix} X' \\ Y' \end{bmatrix} \triangleleft \begin{bmatrix} X'' \\ Y'' \end{bmatrix} \to \begin{bmatrix} A \\ C \end{bmatrix}$$

Contour is Adjoint to Splice

What is a canonical algebra of context on top of a monoidal category?

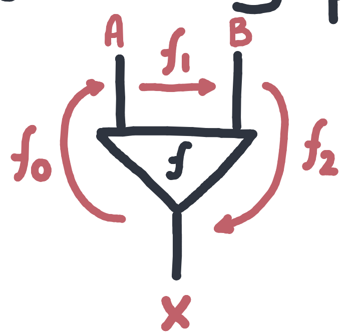
- Each category gives a cofree promonoidal, context.
- Each promonoidal gives a free category, contour.

$$\begin{tikzcd} \text{Category} \arrow[r, bend left, "\text{Context}"] \arrow[r, phantom, "\top"] & \text{Promonoidal Category} \arrow[l, bend left, "\text{Contour}"] \end{tikzcd}$$

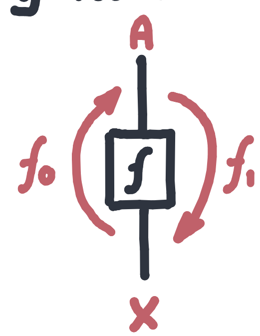
CONTOUR

Meliès, Zeilberger.

Contouring promonoidal categories generates a category.



$$f_0 : X^L \to A^L$$
$$f_1 : A^R \to B^L$$
$$f_2 : B^R \to X^R$$



$$f_0 : X^L \to A^L$$
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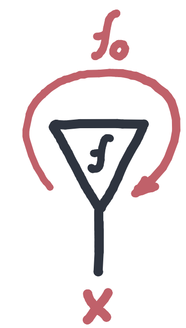
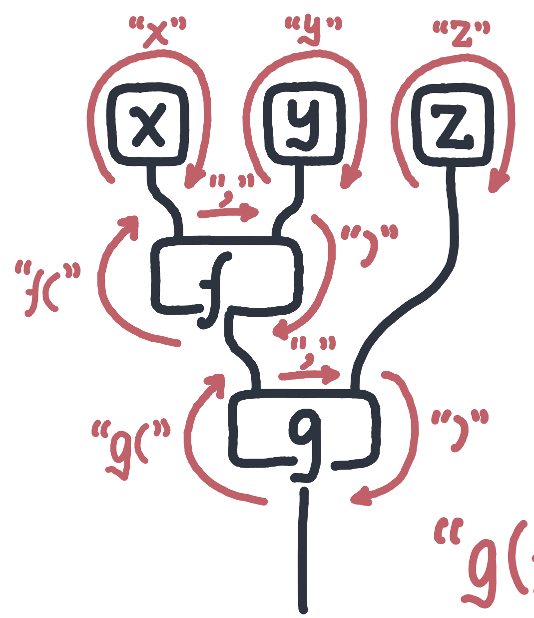


$$f_0 : X^L \to X^R.$$

The category provides
a simple parsing algebra to
any promonoidal,
or any multicategory.



"g(f(x,y),z)"

Next

$$
\begin{tikzcd}
\text{Monoidal Category} \arrow[r, bend left, "\text{Mon. Context}"] \arrow[r, phantom, "\top"] \arrow[d] & \text{Produoidal Category} \arrow[l, bend left, "\text{Mon. Contour}"] \arrow[d] & \\
\text{Category} \arrow[r, bend left, "\text{Context}"] \arrow[r, phantom, "\top"] & \text{Promonoidal Category} \arrow[l, bend left, "\text{Contour}"] & \text{Duoidal Category} \arrow[ul, dashed]
\end{tikzcd}
$$

Part 4 : Context for Monoidal Categories

Monoidal Context-Contour

What is a canonical algebra of decomposition on top of a monoidal category?

- Each monoidal category gives a cofree produoidal, monoidal context.
- Each produoidal gives a free monoidal category, monoidal contour.

$$
\begin{tikzcd}
\text{Monoidal Category} \arrow[r, bend left, "\text{Monoidal Context}"] \arrow[r, phantom, "\top"] & \text{Produoidal Category} \arrow[l, bend left, "\text{Monoidal Contour}"]
\end{tikzcd}
$$

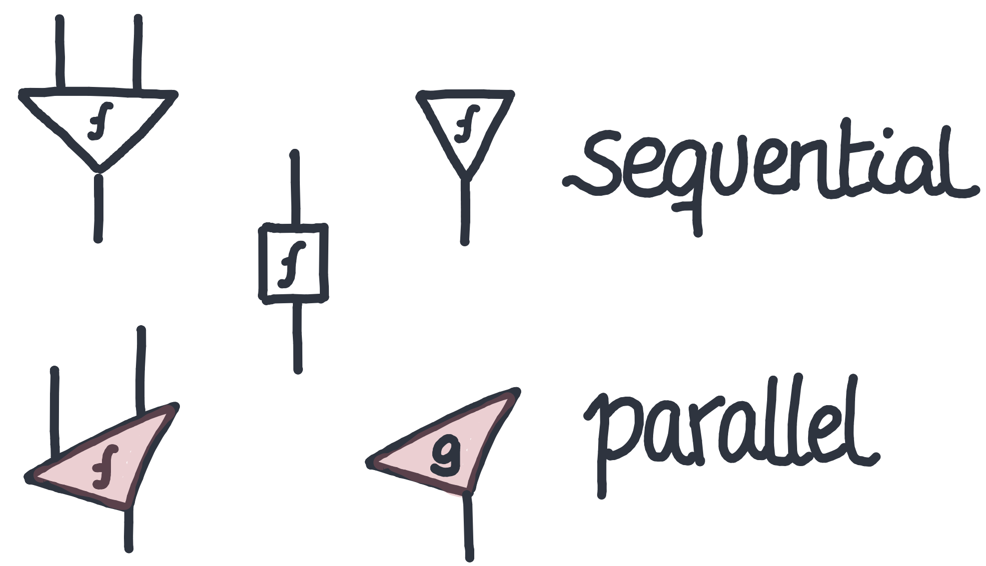
Produoidal Categories

Definition. A produoidal is a pair of promonoidals

$$\mathbb{V}(\bullet \triangleleft \bullet ; \bullet) : \mathbb{V}^{op} \times \mathbb{V} \times \mathbb{V} \to \mathbf{Set}, \quad \mathbb{V}(\bullet ; N) : \mathbb{V}^{op} \to \mathbf{Set}, \quad \text{"sequential"},$$
$$\mathbb{V}(\bullet ; \bullet \otimes \bullet) : \mathbb{V}^{op} \times \mathbb{V} \times \mathbb{V} \to \mathbf{Set}, \quad \mathbb{V}(\bullet ; I) : \mathbb{V}^{op} \to \mathbf{Set}, \quad \text{"parallel"}.$$

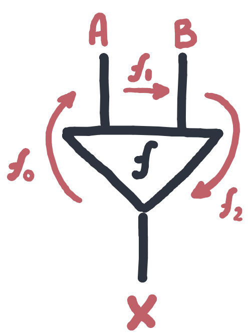
One laxly distributes over the other,

$$\begin{aligned}
\psi_2 &: (A \triangleleft B) \otimes (C \triangleleft D) \longrightarrow (A \otimes C) \triangleleft (B \otimes D), \\
\psi_0 &: I \longrightarrow N \\
\varphi_2 &: N \longrightarrow N \triangleleft N \\
\varphi_0 &: I \longrightarrow I \otimes I
\end{aligned}$$



Monoidal Contour

Contouring produoidal categories generates a monoidal category.



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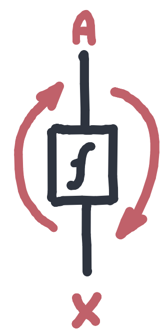

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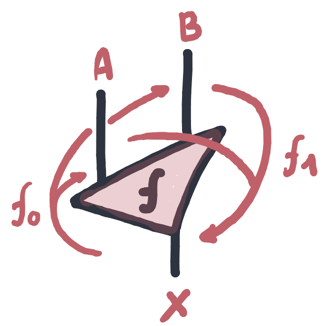

$f_0 : X^L \to A^L \otimes B^L$
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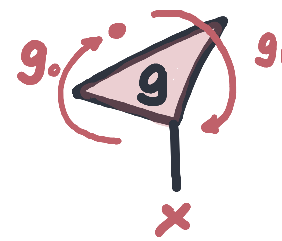
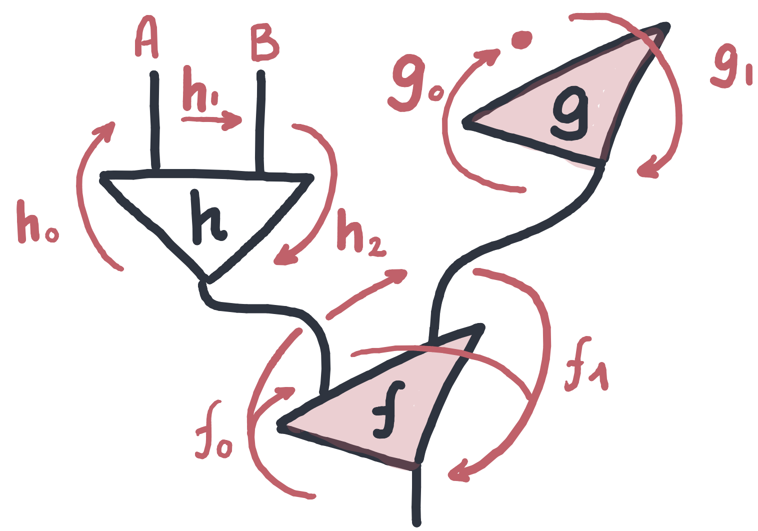
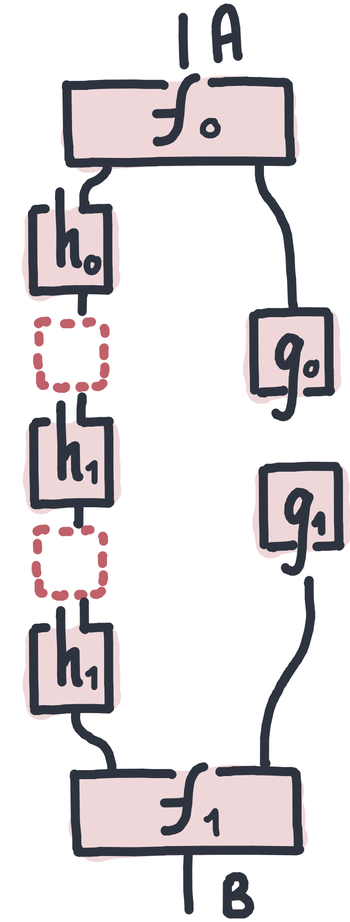


$g_0 : X^L \to I$
$g_1 : I \to X^R$

Monoidal Contour

Contouring produoidal categories generates a monoidal category. Example.



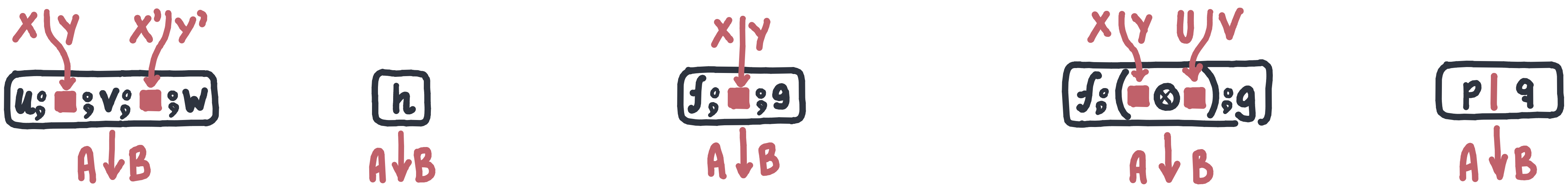


Monoidal Context

Consider 'expressions with holes' in a monoidal category, like the following

$$u ; \blacksquare ; v ; \blacksquare ; w, \qquad k, \qquad f ; (\blacksquare \otimes \blacksquare) ; g, \qquad p \mid q .$$

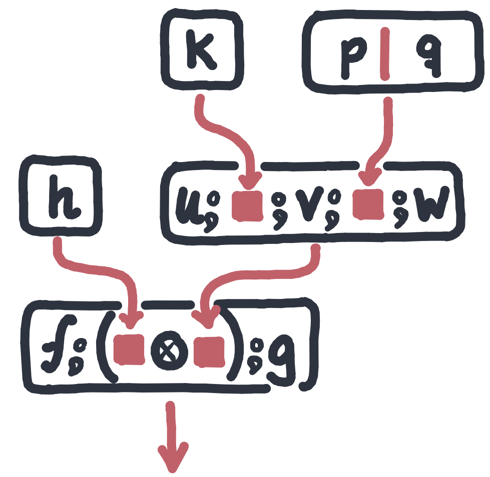
These contexts form a produoidal category.



$$\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix} \qquad N \to \begin{bmatrix}A\\B\end{bmatrix} \qquad \begin{bmatrix}X\\Y\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix} \qquad \begin{bmatrix}X\\Y\end{bmatrix} \otimes \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix} \qquad I \to \begin{bmatrix}A\\B\end{bmatrix}$$

Monoidal Context

$$f ; (h \otimes (u ; k ; v ; p ; q ; w)) ; g \quad =$$



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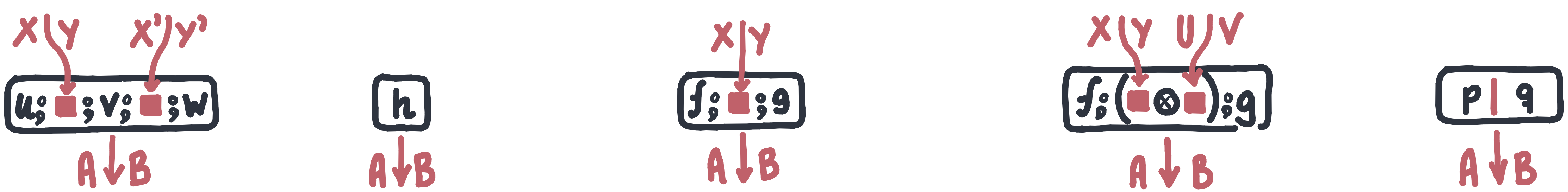

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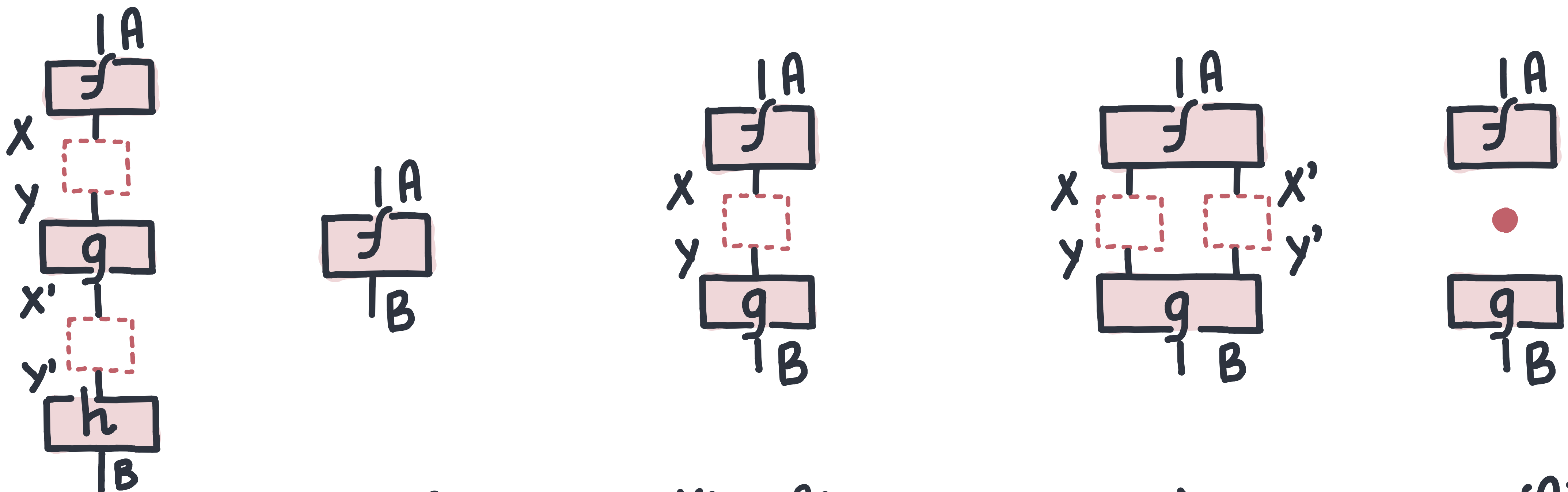
\end{tikzcd}
$$

Monoidal Context

THM (EHR'23). Spliced monoidal arrows are the cofree produoidal on a monoidal.

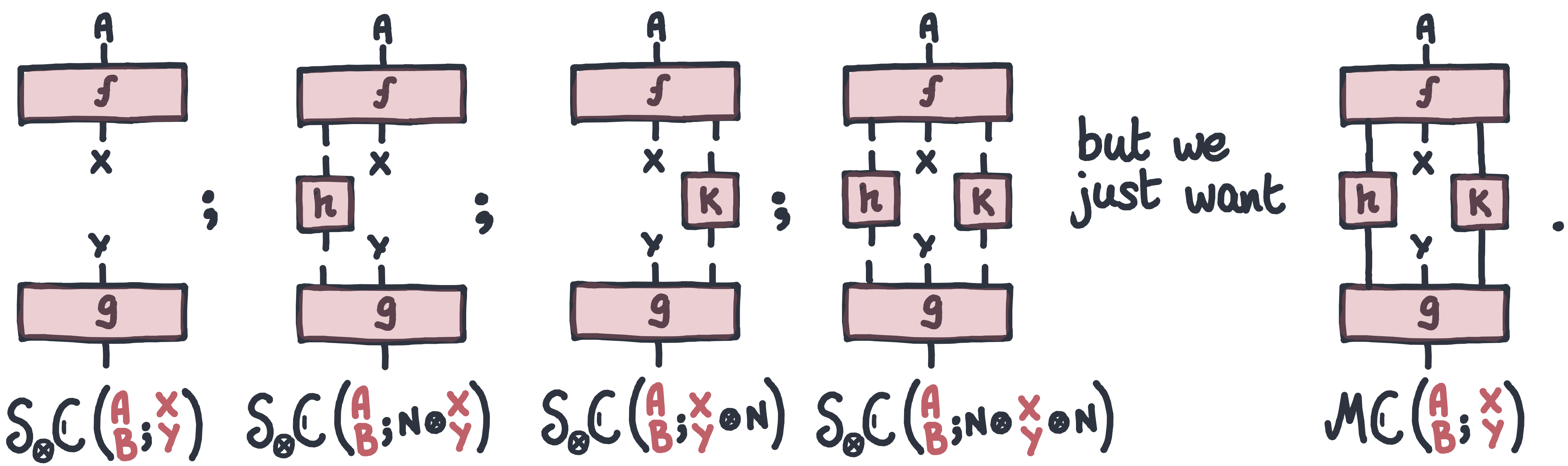


$\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad N \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad \begin{bmatrix}X\\Y\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad \begin{bmatrix}X\\Y\end{bmatrix} \otimes \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad I \to \begin{bmatrix}A\\B\end{bmatrix}$

Missing

Spliced monoidal arrows have some issues:

- They separate sequential and parallel units unnecessarily.
- Produoidals introduce a lot of bureaucracy on units.



PART 5: NORMALIZATION

OPTICS FOR MONOIDAL CATEGORIES

Normalizing Duoidals

A duoidal $(\triangleleft, N, \otimes, I)$ is normal whenever $I \xrightarrow{\cong} N$.

- Being normal is a property (idempotent monad?).
- However, we cannot normalize any duoidal.

THEOREM (Garner, López Franco). Let $(V, \otimes, I, \triangleleft, N)$ a duoidal with reflexive coequalizers, preserved by $(\otimes)$. Then, $(\mathrm{Bimod}^{\otimes}_{N}, \otimes_N, N, \triangleleft, N)$ is a normal duoidal. Similarly for symmetric duoidals.

Garner & López Franco. Commutativity.

Normalizing Produoidals

Theorem (EHR23). We can ALWAYS normalize a produoidal category. Moreover, $\mathrm{Normalization} \colon \mathrm{Produo} \to \mathrm{Produo}$ is an idempotent monad, constructing a free normalization. Similarly for symmetric produoidals.

Every duoidal is indeed normalizable, but the result may be a produoidal.

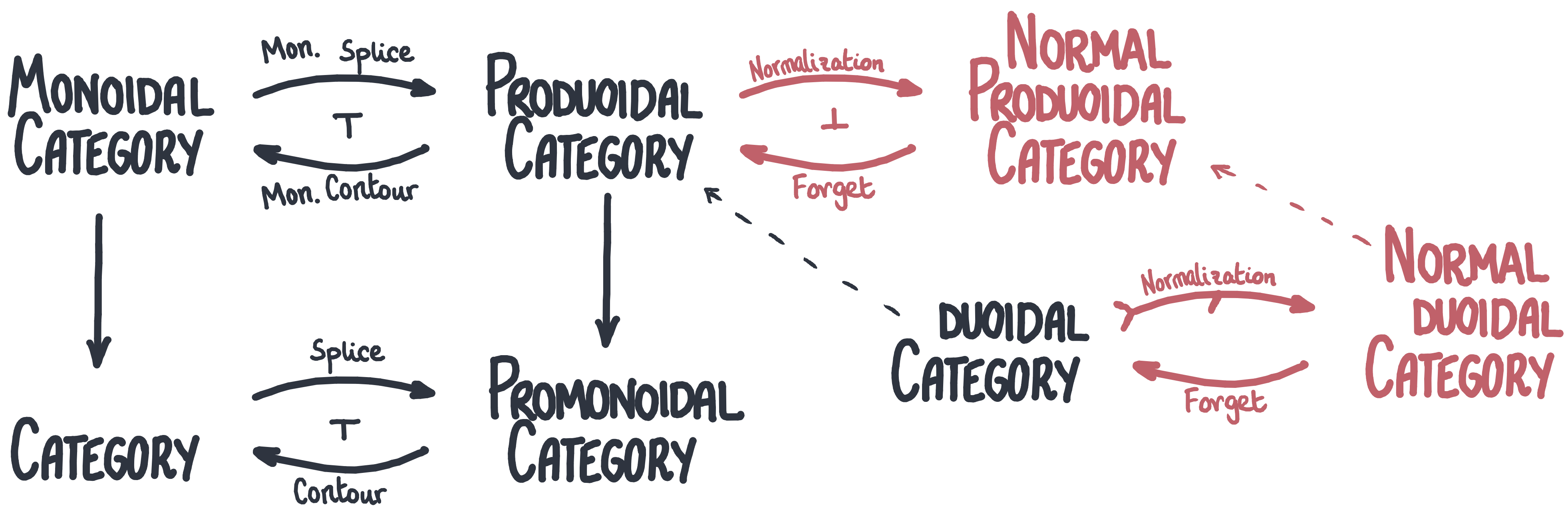
$$\mathcal{N}\mathbb{V}(X;Y) = \mathbb{V}(X; N \otimes Y \otimes N), \qquad \mathcal{N}_{\circ}\mathbb{V}(X;Y) = \mathbb{V}(X; N \otimes Y),$$

$$\mathcal{N}\mathbb{V}(X;Y \triangleleft_N Z) = \mathbb{V}(X;(N \otimes Y \otimes N) \triangleleft (N \otimes Z \otimes N)), \qquad \mathcal{N}_{\circ}\mathbb{V}(X;Y \triangleleft_N Z) = \mathbb{V}(X;(N \otimes Y) \triangleleft (N \otimes Z)),$$

$$\mathcal{N}\mathbb{V}(X;Y \otimes_N Z) = \mathbb{V}(X; N \otimes Y \otimes N \otimes Z \otimes N), \qquad \mathcal{N}_{\circ}\mathbb{V}(X;Y \otimes_N Z) = \mathbb{V}(X; N \otimes Y \otimes Z),$$

$$\mathcal{N}\mathbb{V}(X;N_N) = \mathcal{N}\mathbb{V}(X;I_N) = \mathbb{V}(X; N), \qquad \mathcal{N}_{\circ}\mathbb{V}(X;N_N) = \mathcal{N}_{\circ}\mathbb{V}(X;I_N) = \mathbb{V}(X; N).$$

Next



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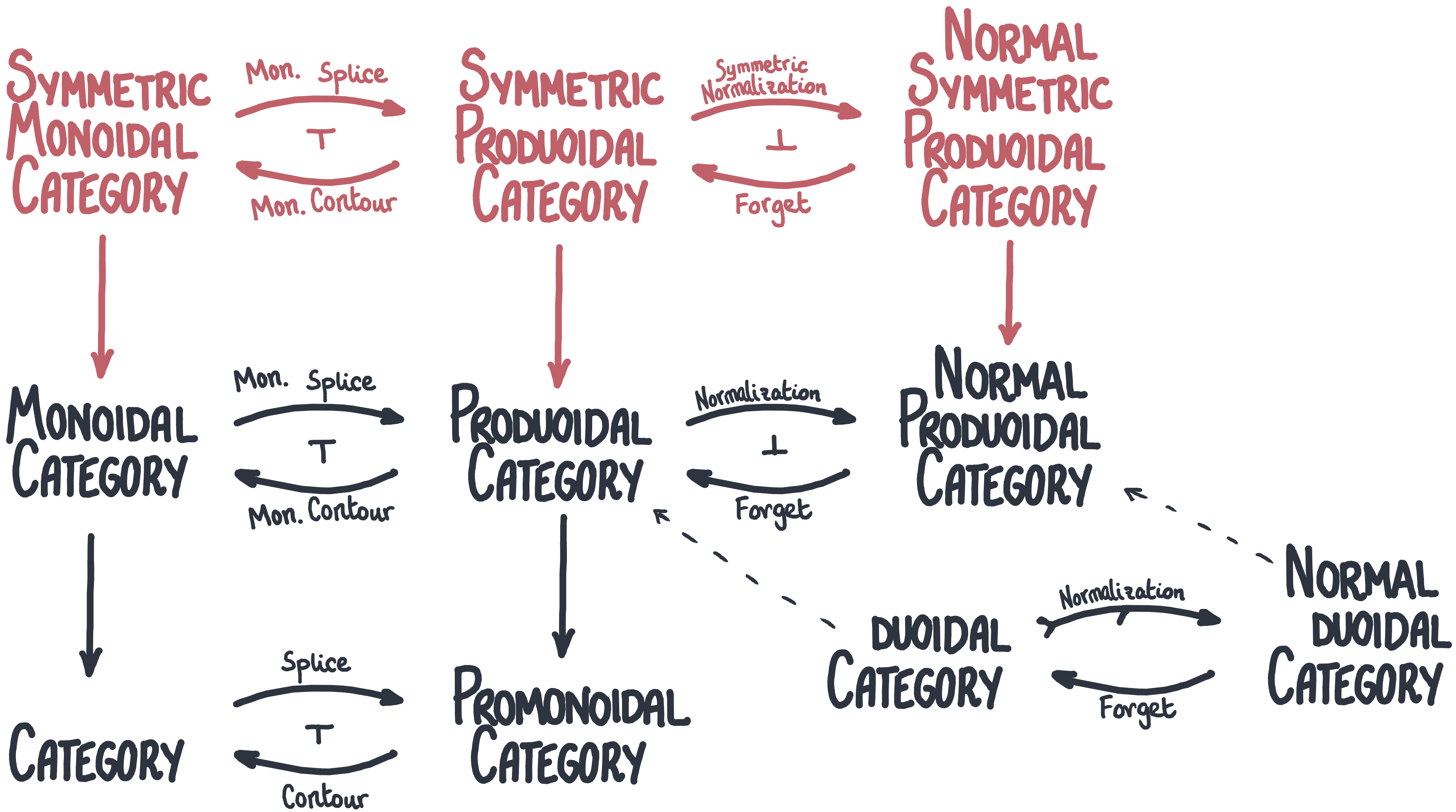
NEXT



Symmetric Monoidal Category
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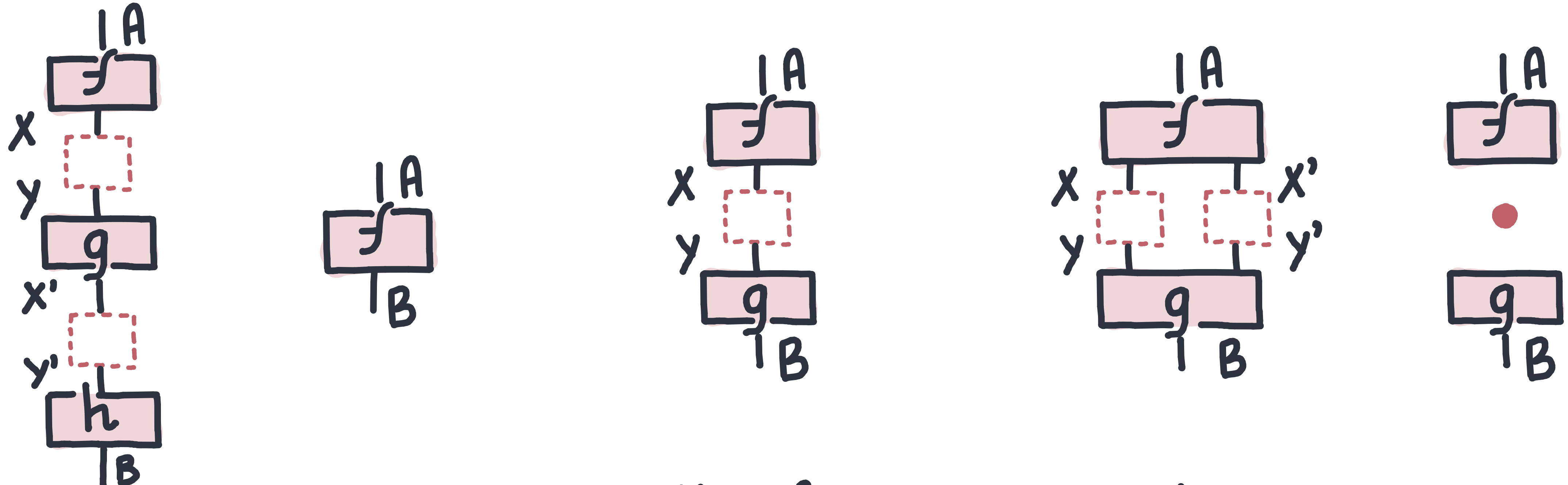
Monoidal Context

THM (EHR'23). Monoidal context is the cofree produoidal on a monoidal.



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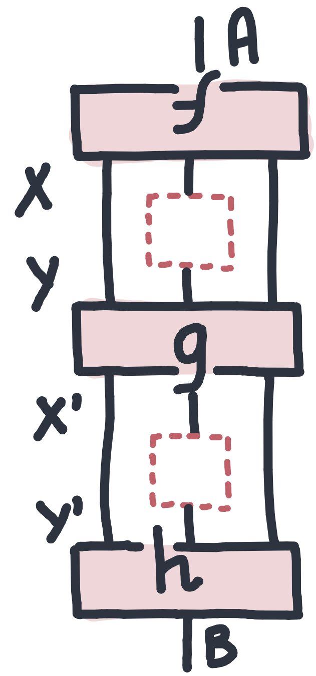
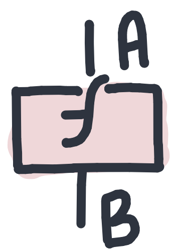
Normalized Monoidal Context

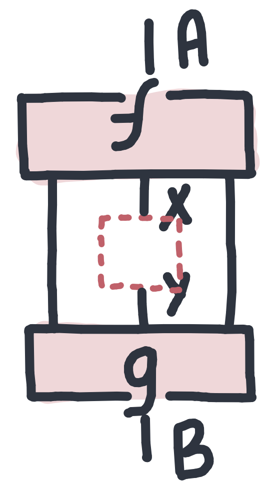
THM (EHR'23). Monoidal optics are the free normalization of monoidal context.



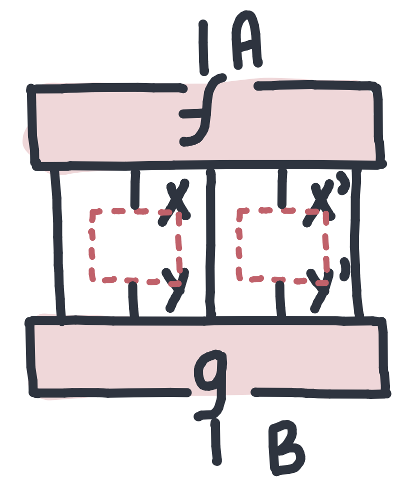
$$\begin{bmatrix} X \\ Y \end{bmatrix} \triangleleft \begin{bmatrix} X' \\ Y' \end{bmatrix} \to \begin{bmatrix} A \\ B \end{bmatrix}$$



$$\begin{bmatrix} A \\ B \end{bmatrix}$$



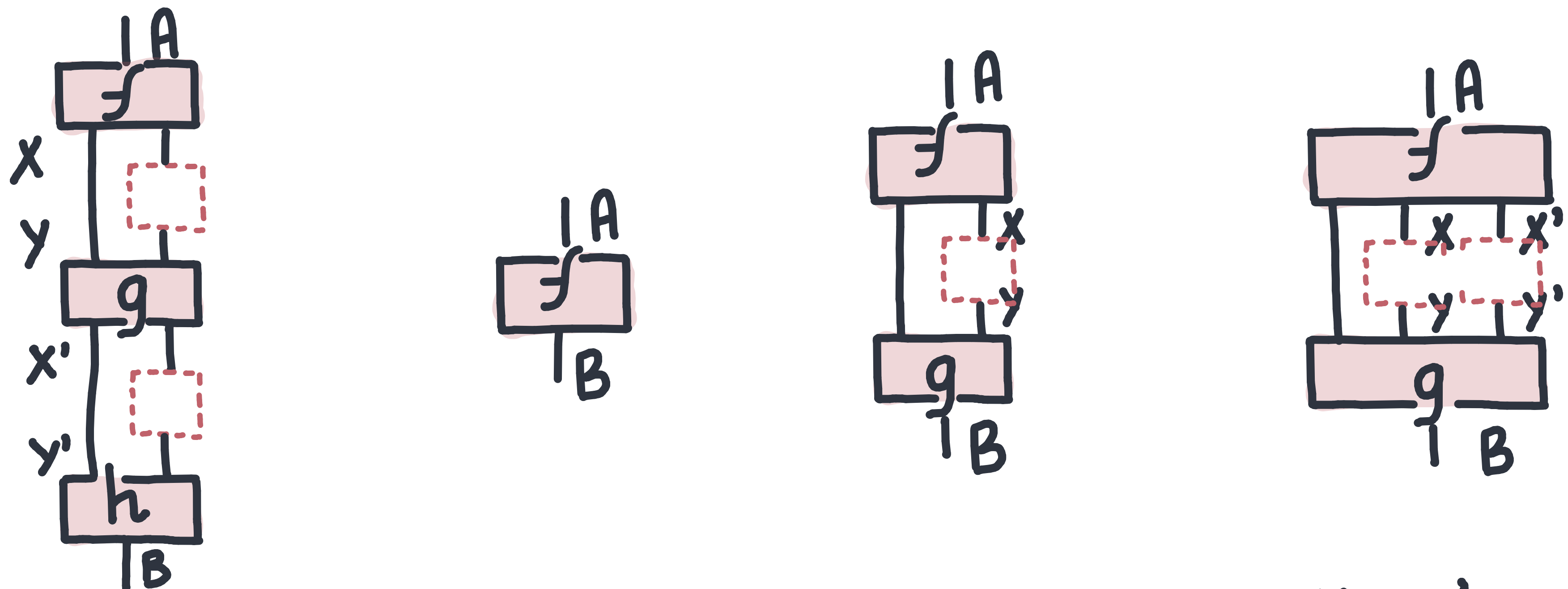
$$\begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} A \\ B \end{bmatrix}$$



$$\begin{bmatrix} X \\ Y \end{bmatrix} \otimes \begin{bmatrix} X' \\ Y' \end{bmatrix} \to \begin{bmatrix} A \\ B \end{bmatrix}$$

Normalized Symmetric Monoidal Context

Thm (EHR'23). Monoidal optics are the free normalization of monoidal context.

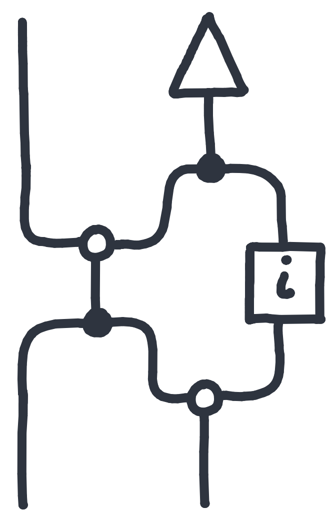


$\begin{bmatrix} X \\ Y \end{bmatrix} \triangleleft \begin{bmatrix} X' \\ Y' \end{bmatrix} \to \begin{bmatrix} A \\ B \end{bmatrix}$ $\quad \begin{bmatrix} A \\ B \end{bmatrix}$ $\quad \begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} A \\ B \end{bmatrix}$ $\quad \begin{bmatrix} X \\ Y \end{bmatrix} \otimes \begin{bmatrix} X' \\ Y' \end{bmatrix} \to \begin{bmatrix} A \\ B \end{bmatrix}$

Part 6: Example

ONE-TIME PAD

Broadbent & Karvonen propose a formalization of the one-time pad in a monoidal category with a Hopf algebra with an integral.



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Broadbent & Karvonen. Categorical Composable Cryptography.

ONE-TIME PAD

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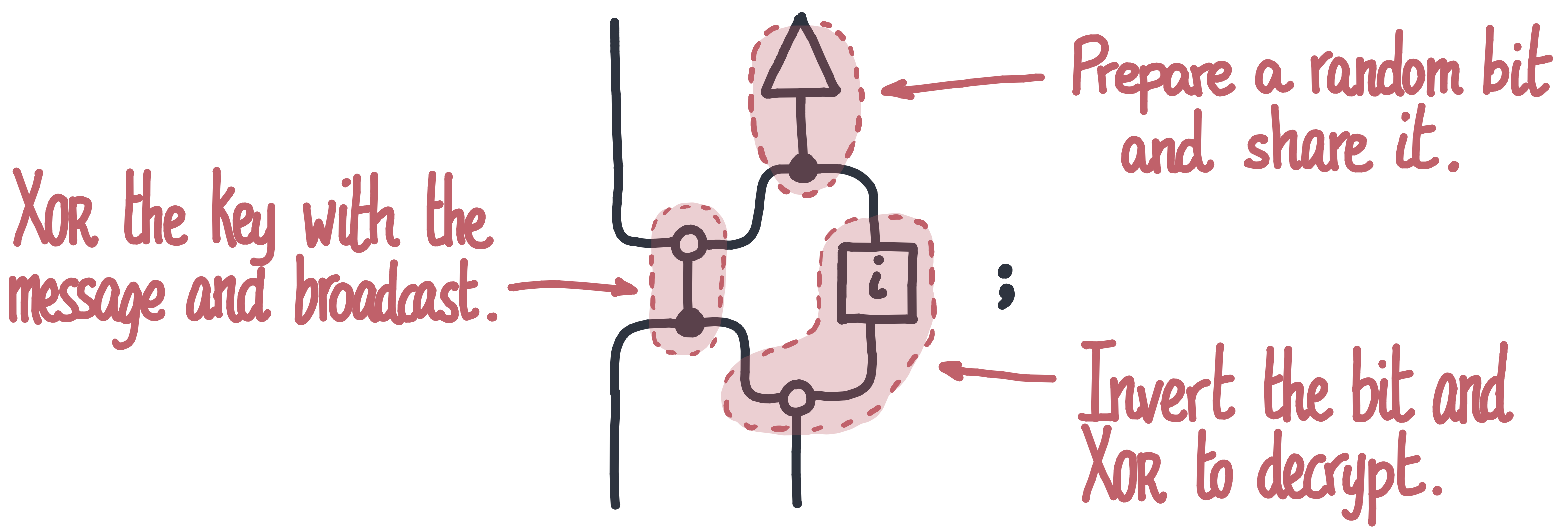

Broadbent & Karvonen. Categorical Composable Cryptography.

One-Time Pad

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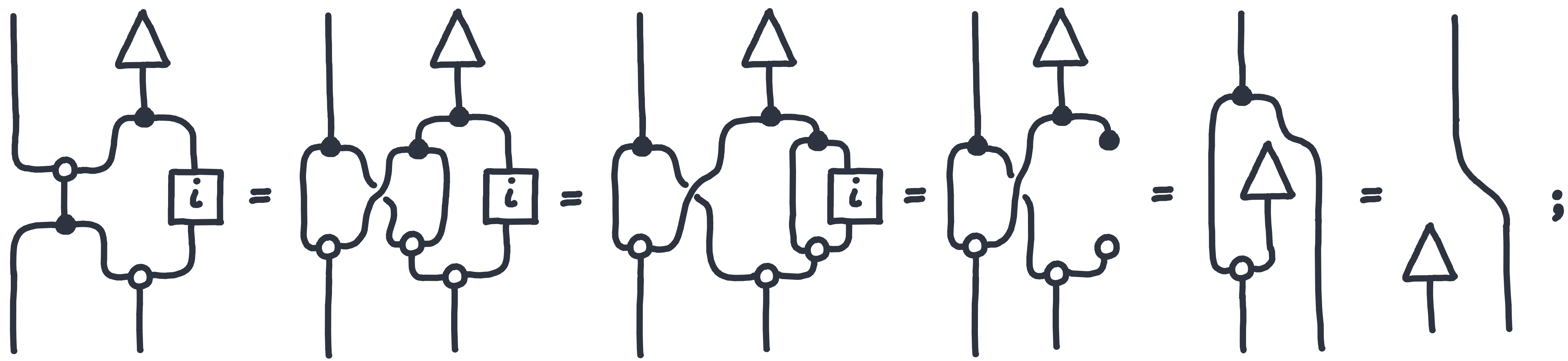
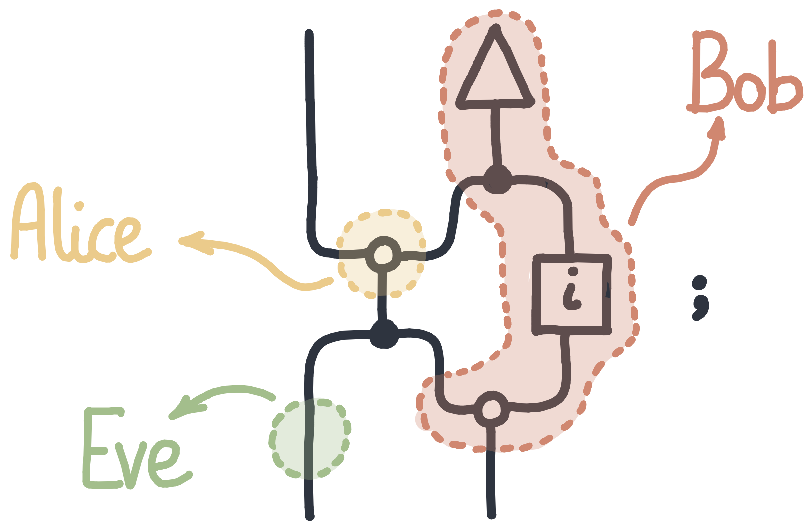


We can reason about security using string diagrams.

Broadbent & Karvonen. Categorical Composable Cryptography.

ONE-TIME PAD

We want to split the morphism into different agents: Alice does not control the broadcast; Eve can only attack at the end; Bob keeps a bit in memory.



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ONE-TIME PAD

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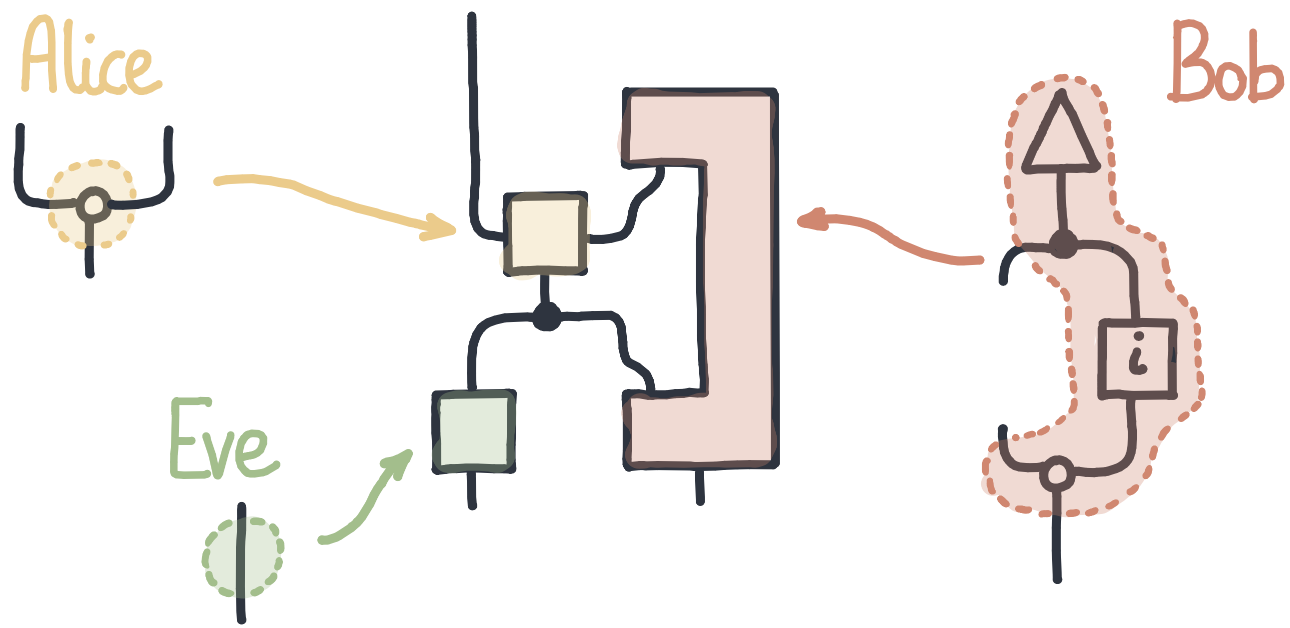


The set of possible actions of Alice and Eve are given by a hom-set; they are monoidal morphisms. What about Bob?

One-Time Pad

This is not only about string diagrams; this is about code modularity and separation.

```
oneTimePad(msg) = do
  key <- randomBit
  crypt <- xor(msg,key)
  msg <- xor(crypt,key)
  return msg
```

Do-notation is a syntax for (pre)monoidal categories; following string diagrams. We can extend it with message-passing, and split into components.

Heunen & Jacobs, Hughes, Staton & Levy, Román.

ONE-TIME PAD

github.com/mroman42/one-time-pad-example

This is not only about string diagrams; this is about code modularity and separation.

```
oneTimePad(alice,bob,eve,msg) = do
  key <- bob₀()
  crypt <- alice(msg, key)
  () <- eve(crypt)
  msg <- bob₁(crypt)
  return msg
```

```
alice(msg, key) = do
  crypt <- xor(msg,key)
  return crypt
```

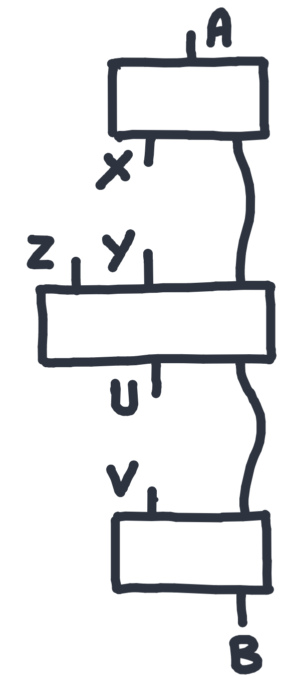
```
eve(crypt) = do
  return crypt
```

```
bob() = do
  key <- randomBit
  !key
  ?crypt
  msg <- xor(crypt,key)
  return msg
```

FURTHER WORK

In the category of lenses, we can write exchanges, e.g.

$$\mathcal{LC}\left(\begin{smallmatrix}A\\B\end{smallmatrix}; \left(\begin{smallmatrix}X\\Y\end{smallmatrix} \otimes \begin{smallmatrix}I\\Z\end{smallmatrix}\right) \triangleleft \begin{smallmatrix}U\\V\end{smallmatrix}\right)$$



PROPOSITION. The $\otimes$ of lenses is representable. Lenses are monoidal with $\binom{X}{Y} \otimes \binom{X'}{Y'} = \binom{X \otimes X'}{Y \otimes Y'}$.

$$\mathcal{LC}\left(\begin{smallmatrix}A\\B\end{smallmatrix}; \begin{smallmatrix}X\\Y\otimes Z\end{smallmatrix} \triangleleft \begin{smallmatrix}U\\V\end{smallmatrix}\right)$$

PROPOSITION. There exist mon. functors $(!) \colon \mathcal{C} \to \mathcal{LC}$ and $(?) \colon \mathcal{C}^{op} \to \mathcal{LC}$. These satisfy $!X = \binom{X}{I}$, $?X = \binom{I}{X}$, with $\binom{X}{Y} = !X \otimes ?Y = !X \triangleleft ?Y$,

! SEND

? RECEIVE

$$\mathcal{LC}\left(\begin{smallmatrix}A\\B\end{smallmatrix}; !X \triangleleft ?(Y \otimes Z) \triangleleft !U \triangleleft ?V\right).$$

Honda et al.

END