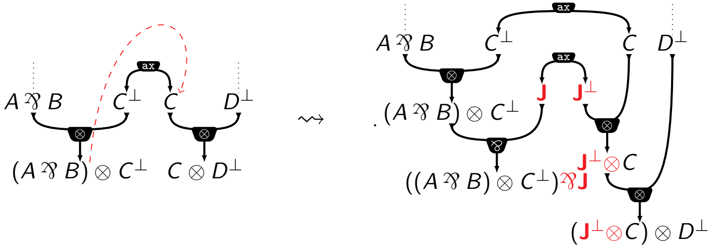
The Exponential Logic of Sequentialization



Aurore Alcolei
LACL, Créteil

Luc Pellissier
LACL, Créteil

Alexis Saurin
IRIF, CNRS, Paris

MFPS XXXIX – June, 22 2023

MLL proofs and nets

MLL $\quad F = F \otimes F \mid F \parr F \mid A \mid A^\perp$

Proof trees

$$\frac{}{\vdash A, A^\perp}\ \mathrm{Ax} \qquad \frac{\vdash \Gamma, F \qquad \vdash \Delta, F^\perp}{\vdash \Gamma, \Delta}\ \mathrm{cut}$$

$$\frac{\vdash \Gamma, F, G}{\vdash \Gamma, F \parr G}\ \parr \qquad \frac{\vdash \Gamma, F \qquad \vdash \Delta, G}{\vdash \Gamma, \Delta, F \otimes G}\ \otimes$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A} \quad \dfrac{}{\vdash B^\perp, B}}{\vdash A^\perp \otimes B^\perp, A, B}\ \otimes}{\vdash A^\perp \otimes B^\perp, A \parr B}\ \parr \quad \dfrac{}{\vdash C^\perp, C}}{\vdash A^\perp \otimes B^\perp, (A \parr B) \otimes C^\perp, C}\ \otimes \quad \dfrac{}{\vdash D^\perp, D}}{\vdash A^\perp \otimes B^\perp, (A \parr B) \otimes C^\perp, C \otimes D^\perp, D}\ \otimes$$

Proof structures

+ Confluence: if $R \Downarrow R'$ then R' is unique (contrary to proof trees)

MLL proofs and nets

MLL $\quad F = F \otimes F \mid F \parr F \mid A \mid A^\perp$

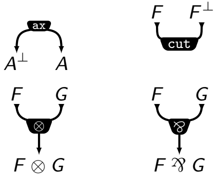
Proof trees

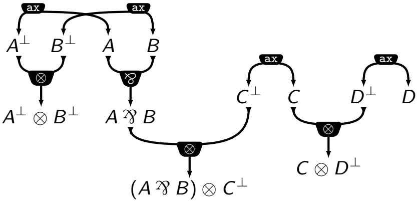
$$\frac{}{\vdash A, A^\perp}\ \mathrm{Ax} \qquad \frac{\vdash \Gamma, F \qquad \vdash \Delta, F^\perp}{\vdash \Gamma, \Delta}\ \mathrm{cut}$$

$$\frac{\vdash \Gamma, F, G}{\vdash \Gamma, F \parr G}\ \parr \qquad \frac{\vdash \Gamma, F \qquad \vdash \Delta, G}{\vdash \Gamma, \Delta, F \otimes G}\ \otimes$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A} \quad \dfrac{}{\vdash B^\perp, B}}{\vdash A^\perp \otimes B^\perp, A, B}\ \otimes}{\vdash A^\perp \otimes B^\perp, A \parr B}\ \parr \quad \dfrac{}{\vdash C^\perp, C}}{\vdash A^\perp \otimes B^\perp, (A \parr B) \otimes C^\perp, C}\ \otimes \quad \dfrac{}{\vdash D^\perp, D}}{\vdash A^\perp \otimes B^\perp, (A \parr B) \otimes C^\perp, C \otimes D^\perp, D}\ \otimes$$

Proof structures





+ Confluence: if $R \Downarrow R'$ then R' is unique (contrary to proof trees)

MLL proofs and nets

MLL $\quad F = F \otimes F \mid F \parr F \mid A \mid A^{\perp}$

Proof trees

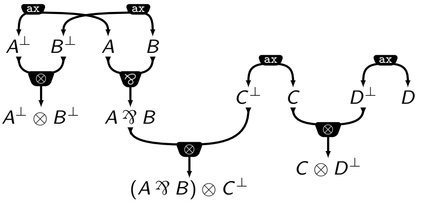
$$\dfrac{}{\vdash A, A^{\perp}}\ \mathrm{Ax} \qquad \dfrac{\vdash \Gamma, F \qquad \vdash \Delta, F^{\perp}}{\vdash \Gamma, \Delta}\ \mathrm{cut}$$

$$\dfrac{\vdash \Gamma, F, G}{\vdash \Gamma, F \parr G}\ \parr \qquad \dfrac{\vdash \Gamma, F \qquad \vdash \Delta, G}{\vdash \Gamma, \Delta, F \otimes G}\ \otimes$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^{\perp}, A} \qquad \dfrac{}{\vdash B^{\perp}, B}}{\vdash A^{\perp} \otimes B^{\perp}, A, B}\ \otimes}{\vdash A^{\perp} \otimes B^{\perp}, A \parr B}\ \parr \qquad \dfrac{}{\vdash C^{\perp}, C}}{\vdash A^{\perp} \otimes B^{\perp}, (A \parr B) \otimes C^{\perp}, C}\ \otimes \qquad \dfrac{}{\vdash D^{\perp}, D}}{\vdash A^{\perp} \otimes B^{\perp}, (A \parr B) \otimes C^{\perp}, C \otimes D^{\perp}, D}\ \otimes$$

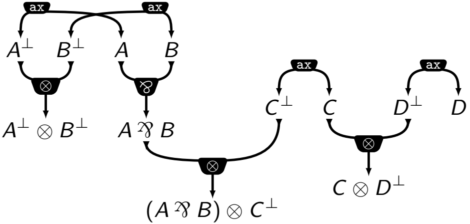
Proof structures

$\downarrow$ deseq



– A proof structure is a net if it is the desequentialisation of some proof tree.

Retrieving some sequentiality: Jumps



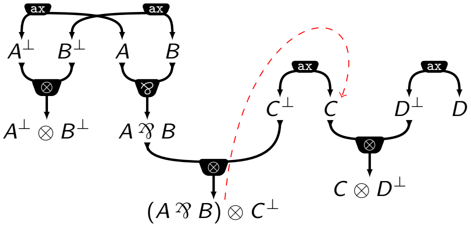
↙ seq ↘

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash A^\perp, A} \qquad \overline{\vdash B^\perp, B}}{\vdash A^\perp \otimes B^\perp, A, B}\,\otimes}{\vdash A^\perp \otimes B^\perp, A \parr B}\,\parr \qquad \overline{\vdash C^\perp, C}}{\vdash A^\perp \otimes B^\perp, (A \parr B) \otimes C^\perp, C}\,\otimes \qquad \overline{\vdash D^\perp, D}}{\vdash A^\perp \otimes B^\perp, (A \parr B) \otimes C^\perp, C \otimes D^\perp, D}\,\otimes$$

$$\dfrac{\dfrac{\dfrac{\overline{\vdash A^\perp, A} \qquad \overline{\vdash B^\perp, B}}{\vdash A^\perp \otimes B^\perp, A, B}\,\otimes}{\vdash A^\perp \otimes B^\perp, A \parr B}\,\parr \qquad \dfrac{\overline{\vdash C^\perp, C} \qquad \overline{\vdash D^\perp, D}}{\vdash C^\perp, C \otimes D^\perp, D}\,\otimes}{\vdash A^\perp \otimes B^\perp, (A \parr B) \otimes C^\perp, C \otimes D^\perp, D}\,\otimes$$

[1991] Quantifiers in linear logic II. Girard.
[2008] Proof nets sequentialisation in multiplicative linear logic. Di Giamberardino, Faggian.

Retrieving some sequentiality: Jumps



↙ seq ↘

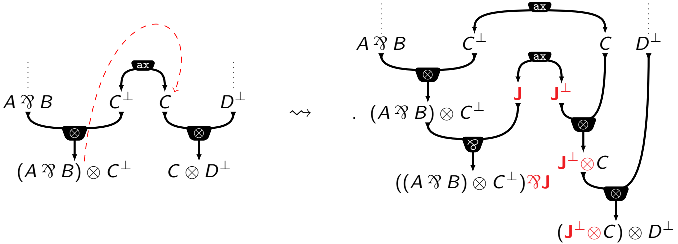
$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash A^\perp, A} \quad \overline{\vdash B^\perp, B}}{\vdash A^\perp \otimes B^\perp, A, B}\otimes}{\vdash A^\perp \otimes B^\perp, A \parr B}\parr \quad \overline{\vdash C^\perp, C}}{\vdash A^\perp \otimes B^\perp, (A \parr B) \otimes C^\perp, C}\otimes \quad \overline{\vdash D^\perp, D}}{\vdash A^\perp \otimes B^\perp, (A \parr B) \otimes C^\perp, C \otimes D^\perp, D}\otimes$$

$$\dfrac{\dfrac{\dfrac{\overline{\vdash A^\perp, A} \quad \overline{\vdash B^\perp, B}}{\vdash A^\perp \otimes B^\perp, A, B}\otimes}{\vdash A^\perp \otimes B^\perp, A \parr B}\parr \quad \dfrac{\overline{\vdash C^\perp, C} \quad \overline{\vdash D^\perp, D}}{\vdash C^\perp, C \otimes D^\perp, D}\otimes}{\vdash A^\perp \otimes B^\perp, (A \parr B) \otimes C^\perp, C \otimes D^\perp, D}\otimes$$

[1991] Quantifiers in linear logic II. Girard.

[2008] Proof nets sequentialisation in multiplicative linear logic. Di Giamberardino, Faggian.

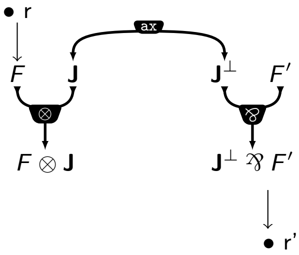
Roadmap: Encoding Jumps



Theorem

$$\begin{array}{ccc} \text{MLL}-\text{structures with jumps} & \xrightarrow{\text{enc}} & \text{MLL}_{\mathbf{J}} - \text{structures} \\ \downarrow \text{seq} & & \downarrow \text{seq} \\ \mathcal{P}(\text{MLL}) & \xleftarrow[\text{dec}]{} & \mathcal{P}(\text{MLL}_{\mathbf{J}}) \end{array}$$

Key Idea: Encoding sequentiality



$\xrightarrow[\text{seq}]{}$

$$\dfrac{\dfrac{\vdots}{\vdash \Gamma, F}\ \mathsf{r} \qquad \dfrac{}{\vdash \mathbf{J}, \mathbf{J}^{\perp}}\ \mathsf{ax}}{\dfrac{\vdash \Gamma, F \otimes \mathbf{J}, \mathbf{J}^{\perp}}{\vdots}}\ \otimes$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \Gamma', F', \mathbf{J}^{\perp}}}{\vdash \Gamma', F' \parr \mathbf{J}^{\perp}}\ \parr}{\vdots}\ \mathsf{r}'$$

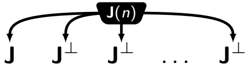
$\Longrightarrow$ r appears above r' in any sequentialization

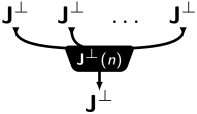
Multiple Jumps

MLL$_\mathbf{J}$ $\quad F_J = A \mid F_J \parr F_J \mid F_J \otimes F_J \mid F_J \parr \mathbf{J}^\perp \mid F_J \otimes \mathbf{J}$

Exponential rules for J

axiom, $n \geq 0$

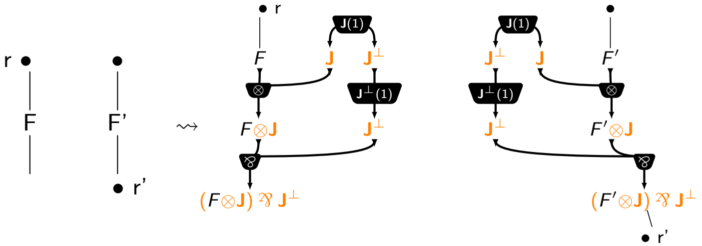




ctr, $n > 0$

Encoding steps:

1) Replacing edges

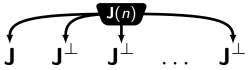


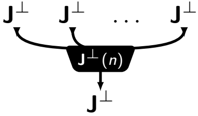
Multiple Jumps

MLLJ $\quad F_J = A \mid F_J \parr F_J \mid F_J \otimes F_J \mid F_J \parr \mathbf{J}^\perp \mid F_J \otimes \mathbf{J}$

Exponential rules for J

axiom, $n \geq 0$

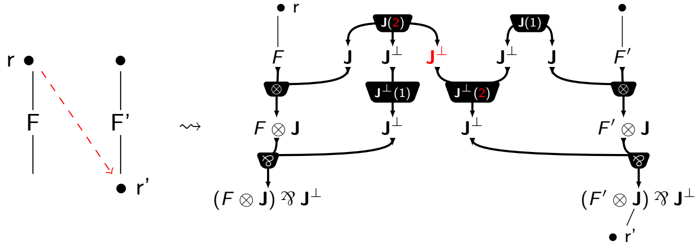




ctr, $n > 0$

Encoding steps:

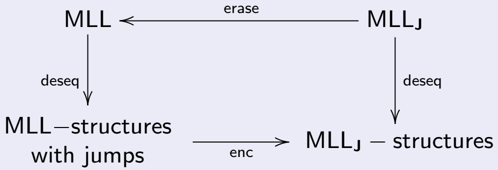
1) Replacing edges
2) Adding jumps



Multiple Jumps

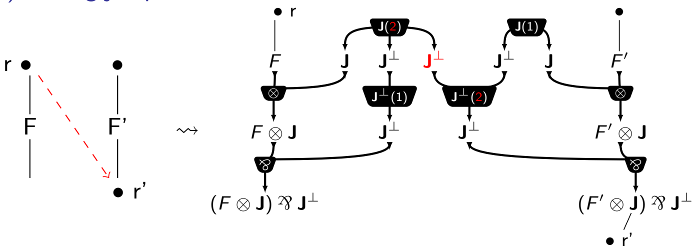
Theorem

Without cuts,



Encoding steps:

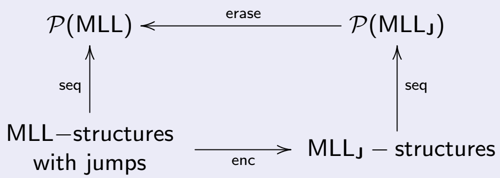
1) Replacing edges
2) Adding jumps



Multiple Jumps

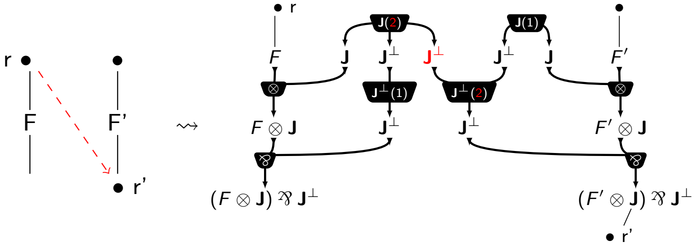
Theorem

Without cuts,



Encoding steps:

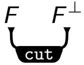
1) Replacing edges
2) Adding jumps



Dealing with cuts

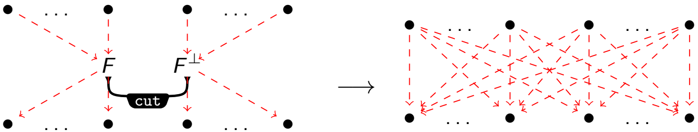
Problems:

1. Retrieving symetries



$\not\leftrightarrow \quad (F \otimes J) \parr J^\perp \, (F^\perp \otimes J) \parr J^\perp$

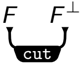
2. Propagating jumps



Dealing with cuts

Problems:

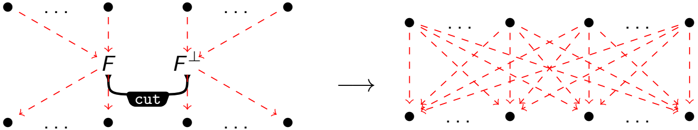
1. Retrieving symetries



$$\not\rightsquigarrow \quad (F \otimes J) \parr J^\perp \, (F^\perp \otimes J) \parr J^\perp$$

$\Longrightarrow$ different gadgets for positive and negative edges

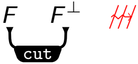
2. Propagating jumps



Dealing with cuts

Problems:

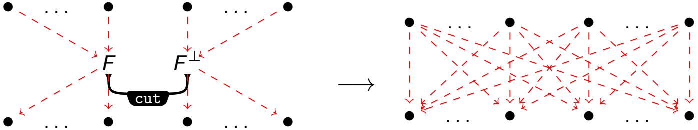
1. Retrieving symetries



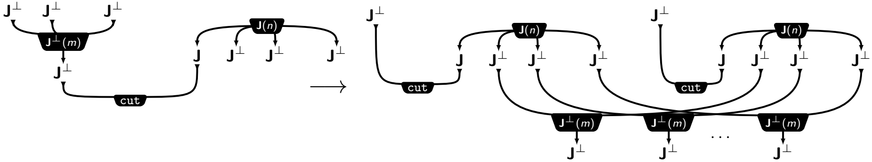
$(F \otimes J) \parr J^{\perp} (F^{\perp} \otimes J) \parr J^{\perp}$

$\Longrightarrow$ different gadgets for positive and negative edges

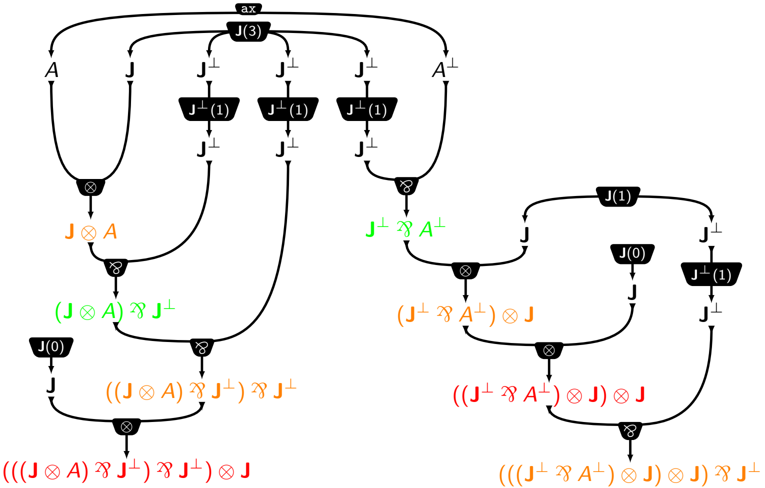
2. Propagating jumps



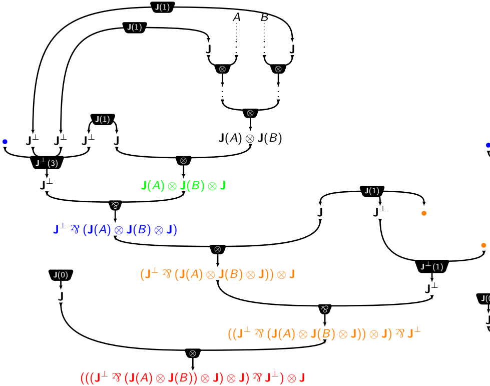
Exponential dynamics

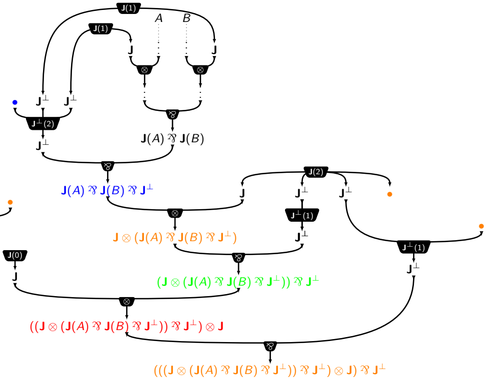


Gadget for Axioms



Gadget for Positive and Negative links





Correctness

Theorem

$$\begin{array}{ccc} \text{MLL}-\text{structures with jumps} & \xrightarrow{\text{enc}} & \text{MLL}_{\jmath} - \text{structures} \\ \Big\downarrow{\scriptstyle \text{seq}} & & \Big\downarrow{\scriptstyle \text{seq}} \\ \mathcal{P}(\text{MLL}) & \xleftarrow[\text{dec}]{} & \mathcal{P}(\text{MLL}_{\jmath}) \end{array}$$

There exists a bisimulation procedure so that the diagram is stable under cut-elimination.

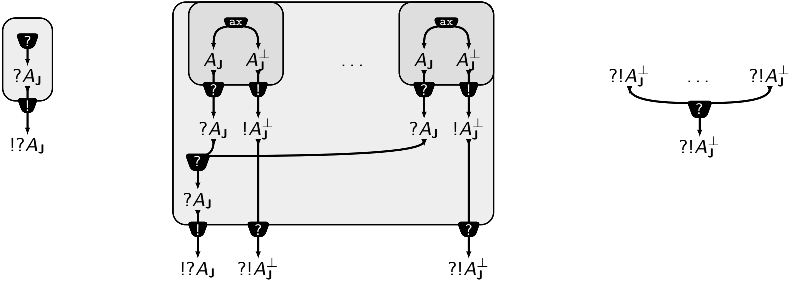
$\rightarrow$ The encoding is linear in the number of links and jumps but cut elimination may induce an exponential growth of the number of jumps.

Encoding $\mathrm{MLL_J}$ in MELL + Mix

Encoding J

$$\mathbf{J} = {!?}A_{\mathbf{J}} \qquad \mathbf{J}^{\perp} = {?!}A_{\mathbf{J}}^{\perp}$$

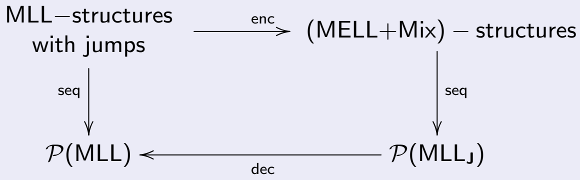
Encoding J rules



✓ Dynamics

The Exponential logic of sequentialization (Conclusion)

Theorem



There exists a bisimulation procedure so that the diagram is stable under cut-elimination.

Conjectures

- Easy extension of the encoding to other fragments of LL
- If $\mathsf{dec}(\mathsf{seq}(\mathsf{enc}(R))) = \{\pi\}$ then $\mathsf{dec}(\mathsf{seq}(\mathsf{enc}(\Downarrow R))) = \{\pi' \mid \pi \Downarrow \pi'\}$