Fractals from Regular Behaviors

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Labelled Transition Systems

A *Labelled Transition System* is A a (finite) set of labels, and (X, α) where X is a set of states and $\alpha : X \to \mathcal{P}(A \times X)$. If $(a, y) \in \alpha(x)$, we usually write $x \stackrel{a}{\to} y$.

We will restrict our attention to *productive* LTSs (in which there are no deadlock states which have no outgoing transitions).

The *traces* emitted by a state x (denoted $tr(x)$) are words $a_1, \ldots, a_n$ from A such that there is a path $x \stackrel{a_1}{\to} x_1 \to \ldots \to x_{n-1} \stackrel{a_n}{\to} x_n$.

x emits a *stream* (denoted $str(x)$) $(a_1, a_2, \ldots)$ of letters from A is for any n, $a_1, \ldots, a_n \in tr(x)$.

In a productive LTS, $str(x) = str(y)$ if and only if $tr(x) = tr(y)$.

Process Terms

$$v \mid ae \mid e_1 + e_2 \mid \mu v\ e$$

We will restrict our attention to closed and guarded (process) terms: all variables v occur within the scope of μv and within the scope of an $a(-)$. (This is a fragment of Milner's fixed point calculus whose terms only specify productive LTSs.)

$$ae \xrightarrow{a} e \qquad \frac{e_1 \xrightarrow{a} f}{e_1 + e_2 \xrightarrow{a} f} \qquad \frac{e_2 \xrightarrow{a} f}{e_1 + e_2 \xrightarrow{a} f} \qquad \frac{e[\mu v\ e/v] \xrightarrow{a} f}{\mu v\ e \xrightarrow{a} f}$$

Figure: The relation $\xrightarrow{a} \subseteq \text{Term} \times \text{Term}$ defining (Term, γ).

Equivalent Terms

$$
\begin{array}{ll}
\text{(ID)} & e + e \equiv e \\
\text{(CM)} & e_2 + e_1 \equiv e_1 + e_2 \\
\text{(AS)} & e_1 + (e_2 + e_3) \equiv (e_1 + e_2) + e_3 \\
\text{(DS)} & a(e_1 + e_2) \equiv ae_1 + ae_2 \\
\text{(FP)} & \mu v \ e \equiv e[\mu v \ e/v]
\end{array}
$$

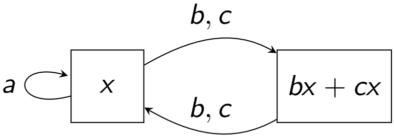
$$
\text{(CN)} \quad \frac{(\forall i)\ e_i \equiv f_i}{g[\vec{e}/\vec{v}] \equiv g[\vec{f}/\vec{v}]}
\qquad
\text{(AE)} \quad \frac{}{\mu w \ e \equiv \mu v \ e[v/w]}
\qquad
\text{(UA)} \quad \frac{g \equiv e[g/v]}{g \equiv \mu v \ e}
$$

$e_1 \equiv e_2$ if they are provably equivalent via these axioms.

Theorem (Rabinovich)

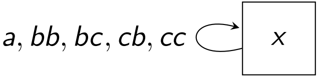
For terms e_1 and e_2, $e_1 \equiv e_2$ if and only if $tr(e_1) = tr(e_2)$.

Example



$$
\begin{aligned}
x &= \mu v(av + b(bv + cv) + c(bv + cv)) \\
&\overset{(FP)}{=} ax + b(bx + cx) + c(bx + cx) \\
&\overset{(DS)}{=} ax + bbx + bcx + cbx + ccx \\
&\overset{(UA)}{=} \mu v(av + bbv + bcv + cbv + ccv)
\end{aligned}
$$

By using labels $\{a, bb, bc, cb, cc\}$, we can see this as a single state LTS.



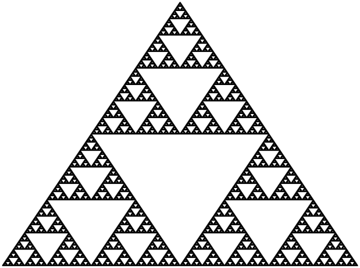
Self Similar Sets (Hutchinson Fractals)

Definition (Hutchinson)

A *(strictly) self-similar set* is the (unique compact) non-empty set-wise fixed point of a finite family of contraction (i.e. distance shrinking) operators on a metric space.

$$\sigma_a \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{2}r + \frac{1}{4} \\ \frac{1}{2}s + \frac{\sqrt{3}}{4} \end{bmatrix} \qquad \sigma_b \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{2}r \\ \frac{1}{2}s \end{bmatrix}$$

$$\sigma_c \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{2}r + \frac{1}{2} \\ \frac{1}{2}s \end{bmatrix}$$



The Sierpiński gasket is the unique non-empty compact subset $\mathbf{S}$ of $\mathbf{R}^2$ such that $\mathbf{S} = \sigma_a(\mathbf{S}) \cup \sigma_b(\mathbf{S}) \cup \sigma_c(\mathbf{S})$.

$$\sigma_a \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{2}r + \frac{1}{4} \\ \frac{1}{2}s + \frac{\sqrt{3}}{4} \end{bmatrix} \qquad \sigma_b \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{2}r \\ \frac{1}{2}s \end{bmatrix}$$

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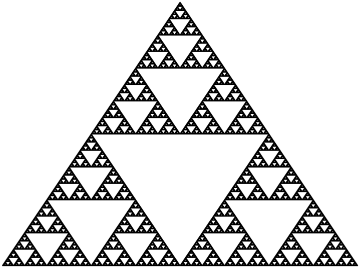


Each point in **S** corresponds to a sequence of a's, b's and c's.
E.g. $(\frac{1}{4}, \frac{1}{2})$ is the limit of the sequence x

$$p, \sigma_b(p), \sigma_b\sigma_a(p), \sigma_b\sigma_a\sigma_a(p), \sigma_b\sigma_a\sigma_a\sigma_a(p), \ldots$$

Self-similar sets are precisely the collection of points obtained in this manner. (Hutchinson)

Systems and Solutions

A *contraction operator interpretation* is a map $\sigma : A \to Con(M, d)$. Given an LTS (X, α), a function $\varphi : X \to \mathbf{K}(M, d)$ (the compact subsets of (M, d)) is a *(σ-)solution* to (X, α) if for any $x \in X$, $\varphi(x) = \bigcup_{x \xrightarrow{a} y} \sigma_a(\varphi(y))$.

Lemma

For a complete metric space (M, d), $\sigma : A \to Con(M, d)$, and (X, α) a finite productive LTS, (X, α) has a unique solution.

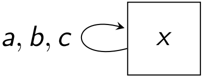
For $x \in X$, the *set generated by x* or *x-component of the solution* is

$$[\![x]\!]_{\alpha,\sigma} = \{\lim_{n\to\infty} \sigma_{a_1} \circ \ldots \circ \sigma_{a_n}(p) : (a_1, a_2, \ldots) \in str(x)\}$$

For a process term e, the *regular subfractal semantics of e* corresponding to σ is $[\![e]\!]_\sigma := [\![e]\!]_{\sigma,\gamma}$.

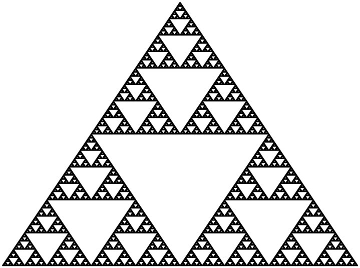
Regular Subfractals

$$\mu v(av + bv + cv)$$



$$\sigma_a \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{2}r + \frac{1}{4} \\ \frac{1}{2}s + \frac{\sqrt{3}}{4} \end{bmatrix} \qquad \sigma_b \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{2}r \\ \frac{1}{2}s \end{bmatrix}$$

$$\sigma_c \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{2}r + \frac{1}{2} \\ \frac{1}{2}s \end{bmatrix}$$



$\sigma_a, \sigma_b, \sigma_c$ is a contraction operator interpretation.

Definition

A *regular subfractal* is the component solution of of a finite productive LTS.

Note: This is a generalization of self-similar sets, self-similar sets are exactly regular subfractals generated by single state LTSs.

Fractal Semantics and Solutions

$$[\![x]\!]_{\alpha,\sigma} = \{\lim_{n\to\infty} \sigma_{a_1} \circ \ldots \circ \sigma_{a_n}(p) : (a_1, a_2, \ldots) \in str(x)\}$$

Theorem

If (X, α) is a finite productive LTS, and (M, d) is a complete metric space, for $x \in X$

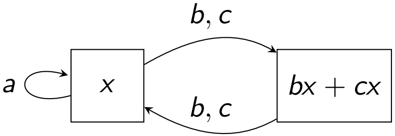
1. *$[\![x]\!]_{\alpha,\sigma}$ is nonempty and compact.*
2. *$[\![-]\!]_{\alpha,\sigma}$ is the unique solution to (X, α).*

Theorem

A subset of a self-similar set is a regular subfractal if and only if it is a component of a solution to a finite productive LTS.

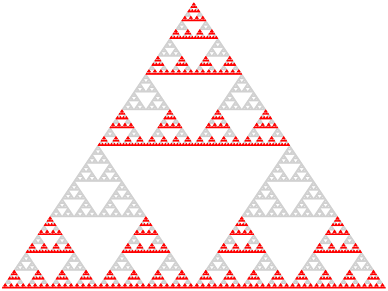
Regular Subfractal Example: Twisted Sierpiński Gasket

$x = \mu v(av + b(bv + cv) + c(bv + cv))$



$$\sigma_a \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{2}r + \frac{1}{4} \\ \frac{1}{2}s + \frac{\sqrt{3}}{4} \end{bmatrix} \qquad \sigma_b \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \frac{r}{2} \\ \frac{s}{2} \end{bmatrix}$$

$$\sigma_c \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \frac{r}{2} + \frac{1}{2} \\ \frac{s}{2} \end{bmatrix}$$



As we saw,
$\mu v(av + b(bv + cv) + c(bv + cv)) \equiv \mu v(av + bbv + bcv + cbv + ccv)$,
which gives a single state LTS, so this *should be* a self similar set.

Soundness and Completeness

For terms e, f, write $e \approx f$ if for every complete metric space (M, d) and every contraction operator interpretation $\sigma : A \to Con(M, d)$, $[\![e]\!]_\sigma = [\![f]\!]_\sigma$. That is, e and f give the same *fractal recipe*.

Theorem

$e \equiv f \Leftrightarrow e \approx f$

Soundness ($\Rightarrow$):

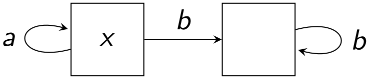
- Uses the fact that LTS homomorphisms preserve traces, so they preserve fractal equivalence, most axioms come from Milner's work.

Completeness ($\Leftarrow$):

- Mainly from Rabinovich and uniqueness of solutions.

Question: Are all regular subfractals self-similar sets?

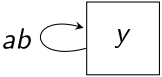
Consider the regular subfractal:



x emits	$[\![x]\!]_{\sigma,\alpha}$
infinite stream of a's	0
infinite stream of b's	1
n many a's followed by an infinite stream of b's	$\frac{1}{2^n}$

Contraction Operation Interpretation: $\sigma_a(x) = \frac{1}{2}x$ and $\sigma_b(x) = \frac{1}{2}x + \frac{1}{2}$

We can obtain $[\![x]\!]_{\sigma,\alpha}$ as the component solution of a single state LTS with a *different* contraction operation interpretation:



$\sigma_a(r) = \frac{1}{2}r$ and $\sigma_b(r) = 1$

However, we can show that $[\![x]\!]_{\sigma,\alpha}$ cannot be obtained from a single state LTS using (finite compositions of) the same contraction operator interpretations (like we did for the Twisted Sierpiński gasket).

Conclusion

Process Terms $\leftrightarrow$ LTS's $\leftrightarrow$ Regular Subfractals
via trace semantics

- Soundness and Completeness: Process terms are provably equivalent if and only if under any contraction operation interpretation, they yield the same fractals.
- We provide a specification language that was not present in the fractals literature; separation of syntax (the process terms) and semantics (the fractals obtained using contraction operator interpretations of the syntax) is something that comes from the tradition of logic and theoretical computer science.

Questions:

- Are regular subfractals identical to self similar sets?
- Since LTSs (and other automata) appear so frequently in decision procedures from process algebra and verification, we would like to know if our semantics perspective on fractals can provide new complexity results in fractal geometry.

Thank you!