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Traced monoidal categories as algebraic structures in Prof

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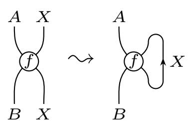
Abstract

We define a traced pseudomonoid as a pseudomonoid in a monoidal bicategory equipped with extra structure, giving a new characterisation of Cauchy complete traced monoidal categories as algebraic structures in **Prof**, the monoidal bicategory of profunctors. This enables reasoning about the trace using the graphical calculus for monoidal bicategories, which we illustrate in detail. We apply our techniques to study traced $*$-autonomous categories, proving a new equivalence result between the left $\otimes$-trace and the right $⅋$-trace, and describing a new condition under which traced $*$-autonomous categories become autonomous.

Keywords: traced, monoidal categories, string diagrams, profunctors

1 Introduction

One way to interpret category theory is as a theory of *systems* and *processes*, whereby monoidal structure naturally lends itself to enable processes to be juxtaposed in parallel. Following this analogy, the presence of a *trace* structure embodies the notion of feedback: some output of a process is directly fed back in to one of its inputs. For instance, if we think of processes as programs, then feedback is some kind of recursion [11]. This becomes clearer still when we consider how tracing is depicted in the standard graphical calculus [16, § 5], as follows:



(1)

Many important algebraic structures which are typically defined as sets-with-structure, like monoids, groups, or rings, may be described abstractly as algebraic structure, which when interpreted in **Set** yield the original definition. We call this process *externalisation.* The external version of a definition can then be reinterpreted in a setting other than **Set** to expose meaningful connections between known structures, or to generate new ones. For instance, a monoid in **Set** is a standard monoid, but a monoid in **Vect** is a unital algebra, and in **Cat** it is a strict monoidal category. Externalisation formalises the relationship between these structures.

In this article, we externalise the 1-categorical notion of traced monoidal category, giving a new external definition of *traced pseudomonoid.* We show that, when interpreted in **Prof**, the monoidal bicategory of categories and profunctors, this is equivalent to the standard definition of traced monoidal category. While the traditional definition of traced monoidal category has five separate axiom families (see Appendix A), our traced pseudomonoid only has three, because

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two of the axioms become subsumed into the technology of **Prof**. In this sense our externalised theory is a simpler than the traditional approach.

We make substantial use of the graphical calculus for compact closed bicategories, categorifying the way one might use a PROP when working in a symmetric monoidal theory [13]. **Prof** additionally admits a special string diagram calculus of *internal string diagrams* — string diagrams 'inside' string diagrams — which we use extensively to prove our results.

We apply our framework to derive new proofs of known facts about traced monoidal categories in an entirely diagrammatic and synthetic way. For instance, every braided autonomous category admits a trace, which we reduce to the presence of a certain isomorphism. Following this, we proceed to analyse the interaction between tracing and $*$-autonomous structure. We show that on a $*$-autonomous category, a right $\otimes$-trace and a left $⅋$-trace are equivalent. We also derive an interesting sufficient condition for a traced $*$-autonomous category to be compact closed, extending previous work of Hajgató and Hasegawa [10] which handled the symmetric case.

1.1 Related work

Our traced pseudomonoid is a sort of *categorification* of the standard categorical notion of trace, as described by Joyal, Street and Verity [12]. The idea of bicategories [5] as a formal arena for the study of categories comes from Gray [9], however an issue which arises is that the obvious arena **Cat** preserves too little information to study certain phenomena. Profunctors are one way to resolve this [19], and furthermore they also naturally allow for the diagrammatic methods we wish to employ. **Prof** is to **Cat** what **Rel** is to **Set** Loregian [14, Example 5.1.5].

Within the same framework, certain Frobenius pseudomonoids categorify the notion of $*$-autonomous categories [1], as first studied by Street [18]. We use the term '$*$-autonomous' for the non-symmetric version, as described by Barr [2]. In a symmetric monoidal category, the notions of trace and $*$-autonomous interact: a traced symmetric $*$-autonomous monoidal category is compact closed [10]. An obvious conjecture is that a traced $*$-autonomous category is autonomous (with left and right duals for all objects), and in the last section of our paper we give an analysis of this problem, deriving a sufficient condition for this result to hold.

1.2 Outline

In Section 2, we establish our technical background, utilising the language of *presentations* [15, § 2.10] to graphically represent different types of monoidal categories. Presentations extending a pseudomonoid with right adjoints represent Cauchy complete monoidal categories when interpreted in **Prof**, and internal string diagrams [4, § 4] are also recalled from existing literature. Section 3 contains our main definition: the traced pseudomonoid presentation. We show that its representations correspond exactly to Cauchy complete traced monoidal categories [12], using internal string diagrams as our main proof technique. Section 4 illustrates using this framework that all Cauchy complete braided autonomous categories are Cauchy complete traced. Section 5 concludes with a study on $*$-autonomous categories, defined by the right-adjoint Frobenius pseudomonoid presentation [8, § 2.7], and their interaction with tracing. We conjecture that every traced $*$-autonomous category is autonomous, which is the non-symmetric generalisation of the result of Hajgató and Hasegawa [10], and use our techniques to give evidence for this conjecture.

1.3 Acknowledgements

The authors are grateful to Masahito Hasegawa for useful comments, and the authors of Bartlett et al. [4] for their TikZ code for drawing internal string diagrams. The first author acknowledges funding from the EPSRC [grant number EP/R513295/1].

2 String diagrams and the bicategory of profunctors

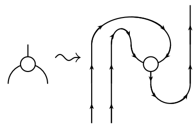
2.1 Introduction

In this section, we establish the definition of **Prof**, and recall some of its important properties. We also assume familiarity with *string diagrams* for compact closed categories, of the type described by Selinger [16, § 4.8]. There are two main differences with our string diagrams:

(i) our string diagram convention is from bottom to top, rather than left to right;

(ii) our setting is *bicategorical*, which we view in projection. This means that, as usual, 0-morphisms are represented by wire *colourings*, and 1-morphisms are represented by 2-dimensional *tiles* with some number of incoming wires and some number of outgoing wires, but in addition there are 2-morphisms which are represented by wire-boundary-preserving (globular) *rewrites* which act locally. In this context, the equational theory states that certain sequences of rewrites agree, and for each tile there is a 'do nothing' rewrite corresponding to the identity 2-morphism.

This is the diagrammatic calculus of Bartlett [3] enhanced with compact structure, which means that 1-morphisms may be rotated, changing the orientation of their wires appropriately:



Sometimes we will use colour-coded boxes to signal the local site at which a 2-morphism is being applied to aid the reader (for an example, see Definition 2.4.) Additionally, we often use the same symbol to denote a 2-morphism, its inverse, or its adjoint mate; context will disambiguate, but e.g. any 2-morphism labelled α is morally the associator move which type-checks, without significant additional nuance.

Definition 2.1 [Bicategory of Profunctors [6, Proposition 7.8.2]] **Prof** is the bicategory of categories, profunctors, and natural transformations.

Additionally, **Prof** is compact closed in the sense of Stay [17, § 2], with the dual of $\mathcal{C}$ given by $\mathcal{C}^{\mathrm{op}}$; the structural information of this bicategory (the identity profunctor, the symmetry, the co/unit of the compact structure, etc.) is given by variations on the Hom-profunctor $\mathcal{C}(-,=): \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$.

There exists an embedding theorem for $\mathbf{Cat} \to \mathbf{Prof}$.

Lemma 2.2 *For each functor $F: \mathcal{C} \to \mathcal{D}$, there are associated profunctors $F_*: \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$ and $F^*: \mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ given by right and left actions on the Hom-functor $\mathcal{D}(-,=)$:*

$$F_*(d,c) = \mathcal{D}(Fd,c), \quad F^*(c,d) = \mathcal{D}(c,Fd).$$

Either mapping extends to an injective fully faithful pseudofunctor [6, Proposition 7.8.5]. Furthermore, $F^* \dashv F_*$ in **Prof** *[6, Proposition 7.9.1].*

F^* is called the covariant embedding of F, or also its *representation*, and F_* is called the contravariant embedding, or alternatively its *corepresentation*.

Lemma 2.2 justifies the following condition, which we shall make heavy use of throughout.

Theorem 2.3 ([6, Theorem 7.9.3]) *Given a small category $\mathcal{C}$, the following conditions are equivalent:*

(i) *$\mathcal{C}$ is Cauchy complete;*

(ii) *for every small category $\mathcal{D}$, a profunctor $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to$* **Set** *has a right adjoint if and only if it is isomorphic to the covariant embedding of a functor.*

Informally, this describes when a profunctor is 'the same' as a functor. More precisely, it allows us to capture the conditions where we can treat functors as profunctors (and conversely) — when some profunctor P is (isomorphic to) the representation of some functor F — i.e. it justifies the move from doing formal category theory in **Cat** to **Prof**. Thus we must qualify that throughout this article, our object of study is Cauchy complete categories. Note that every category admits a universal embedding into its Cauchy completion via the Karoubi envelope construction [7].

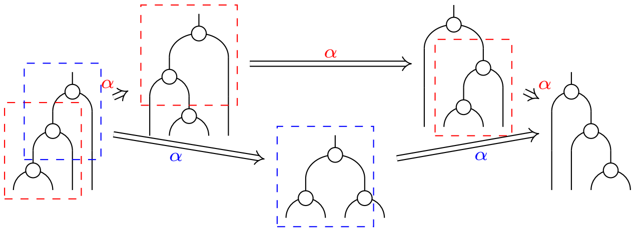
2.2 Planar monoidal categories

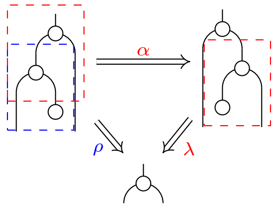
We recall the definitions of the pseudomonoid presentation and its right-adjoint analogue [8].

Definition 2.4 The *pseudomonoid presentation* of $\mathcal{M} = (\cdot,$ , $)$ [3] is given by

- a generating 0-morphism: $\cdot$ [4];
- generating 1-morphisms:  and ;
- invertible generating 2-morphisms expressing associativity and unitality respectively:

- equations witnessing that these inverses are coherent (pentagon [5] and triangle equations):





We actually use *oriented* string diagrams, in the sense of [16, § 4], as the dual of $\cdot$, is given by the duality of $\mathcal{C}$ versus $\mathcal{C}^{\text{op}}$ in **Prof** and is represented diagrammatically by downwards-oriented strings. For cleanliness, we omit decorations for upwards-oriented strings.

Presentations can be interpreted in a target symmetric monoidal bicategory, as follows.

Definition 2.5 An *interpretation* of a presentation $\mathcal{P}$ in a symmetric monoidal bicategory $\mathcal{C}$ is given by a strict symmetric monoidal 2-functor from the free symmetric monoidal bicategory on $\mathcal{P}$ to $\mathcal{C}$.

As discussed in Bartlett et al. [4, § 2.1], such an interpretation corresponds exactly to choosing for each k-dimensional generator of $\mathcal{P}$ a corresponding k-morphism of $\mathcal{C}$, satisfying the corresponding equations. So interpretations of presentations are easy to work with. The following then follows.

Lemma 2.6 *Interpretations of $\mathcal{M}$ in* **Prof** *correspond to Cauchy complete promonoidal categories.*

A promonoidal category is 'nearly' a monoidal category: it captures only when Hom-sets have the form $\mathcal{C}(X, Y \otimes Z)$ — that is, $\otimes$ may only appear as a right action on the Hom. To overcome this limitation, we must restrict our attention to *representable* profunctors (equivalently, profunctors which admit right adjoints).

Definition 2.7 [Free right-adjoint extension] For a presentation $\mathcal{P}$, we denote $\mathcal{P}^{\mathsf{RA}}$ as the presentation with all of the data of $\mathcal{P}$, and in addition, for each generating 1-morphism of $\mathcal{P}$: a

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[3] $\mathcal{M}$ is not a bicategory, rather it is data from which a free symmetric monoidal bicategory can be generated à la generators-and-relations.

[4] The point $\cdot$ represents a 0-dimensional aspect of our graphical calculus, i.e. the 'colour' of the wires at the boundaries of 2-dimensional tiles. Graphically, this 'colour' is depicted by (implicitly) upwards flowing wires, contrasting with its dual colour, the downwards flowing wires.

[5] Due to the weak interchange structure of **Prof**, this pentagon should technically be a hexagon where along the bottom, the right monoid and the left monoid are interchanged between the two associator moves, however for clarity we elide trivial interchange steps throughout.

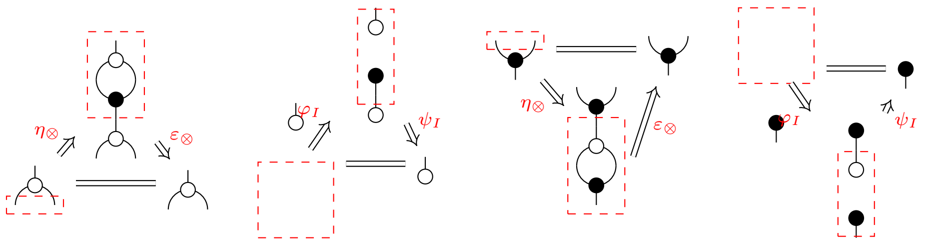
freely-added right-adjoint 1-morphism, unit and counit 2-morphisms, and equational structure witnessing that the triangle equations for this adjunction hold.

The procedure $(-)^{\mathsf{RA}}$ which freely adds right adjoints is well-behaved in the sense of Bartlett et al. [4, § 2.3], and in general, given the data of $\mathcal{P}$, it is unambiguous to discuss a presentation $\mathcal{P}^{\mathsf{RA}}$ with freely-added right adjoints without giving an explicit description as we do in Example 2.8.

Example 2.8 The *right-adjoint pseudomonoid presentation* $\mathcal{M}^{\mathsf{RA}}$ is given by the data of $\mathcal{M}$, and additionally:

- 1-morphisms: and ;
- unit and counit 2-morphisms witnessing adjunctions $\dashv$ and $\dashv$:

- equations witnessing that the adjunction is coherent (triangle equations):

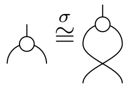


Lemma 2.9 *In the free monoidal bicategory on $\mathcal{M}^{\mathsf{RA}}$, $(\cdot, ,)$ can be given a canonical pseudo-comonoid structure, by transporting the pseudomonoid $(\cdot, ,)$ across the adjunctions.*

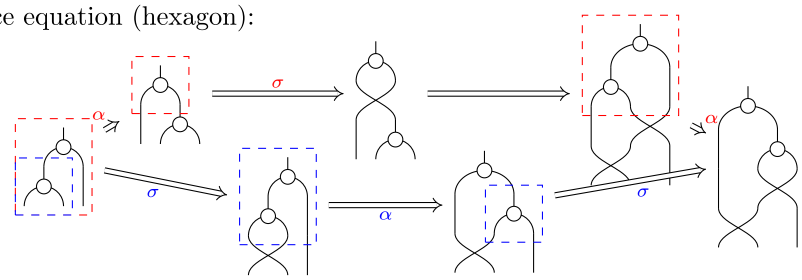
Lemma 2.10 *Interpretations of $\mathcal{M}^{\mathsf{RA}}$ in* **Prof** *correspond to Cauchy complete monoidal categories.*

2.3 Braiding and symmetry

Definition 2.11 The *braided pseudomonoid presentation* $\mathcal{B}$ is given by the data of $\mathcal{M}$, and an additional 2-morphism specifying that the pseudomonoid is commutative:



and coherence equation (hexagon):



Lemma 2.12 *Interpretations of $\mathcal{B}^{\mathsf{RA}}$ in* **Prof** *correspond to Cauchy complete braided monoidal categories.*

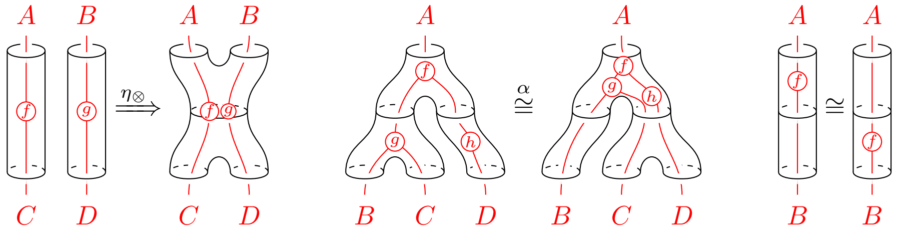
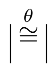


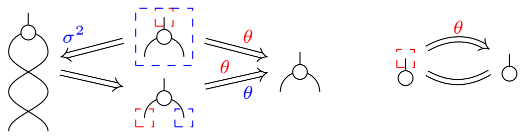
Figure 1. Examples of the internal string diagram formalism.

Likewise, 'braided' can be promoted to 'balanced' by adding a compatible twist in the presentation.

Definition 2.13 The *balanced pseudomonoid presentation* $\mathcal{L}$ is given by the data of $\mathcal{B}$, and additionally a 2-endomorphism specifying a compatible twist:



and equations:



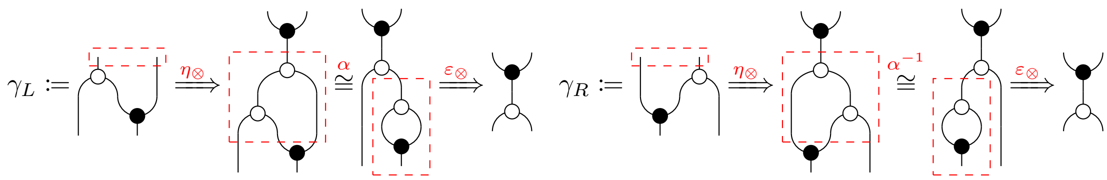
Remark 2.14 The coherence equation of Definition 2.11 is redundant in the presence of a twist. Conversely, the symmetric pseudomonoid presentation is equivalent to the balanced pseudomonoid presentation with a trivial twist.

Subsequently, we shall examine a variety of presentations, representing different types of monoidal categories, which extend $\mathcal{M}$: in each case, their braided (resp. balanced) variant is obtained by considering the corresponding extension with respect to $\mathcal{B}$ (resp. $\mathcal{L}$) instead.

2.4 Autonomous categories

A monoidal category is autonomous when every object has a left and a right dual. Here we recall how they can be defined via a presentation following Bartlett et al. [4].

Definition 2.15 The *autonomous pseudomonoid presentation* $\mathcal{A}$ is given by the data of $\mathcal{M}^{\mathsf{RA}}$, and additionally inverses for the following composite 2-morphisms:

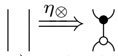


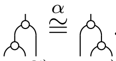
Lemma 2.16 ([4, Proposition 4.8]) *Interpretations of $\mathcal{A}$ in* **Prof** *correspond to Cauchy complete autonomous categories.*

2.5 Internal string diagrams

One special aspect of **Prof** is that it admits a calculus called the *internal string diagram* construction, when we consider presentations which extend $\mathcal{M}^{\mathsf{RA}}$. Informally speaking, the strings we use can be inflated into tubes, containing a volume in which the standard graphical calculus for monoidal categories operates, and 2-morphisms in **Prof** correspond to rewrites of these tubes which act on the internal strings.

Some examples of the formalism are shown in Figure 1. A feature of the formalism is that the internal strings must be read in the opposite direction to the ambient profunctors. Since our profunctor convention is bottom-to-top, the convention for internal strings is top-to-bottom.

Example 2.17 Figure 1(a) illustrates the action of  on internal strings. As a function of sets it maps $\mathcal{C}(A,C) \times \mathcal{C}(B,D) \to \mathcal{C}(A \otimes B, C \otimes D)$, which sends $(f,g) \mapsto f \otimes g$.

In Figure 1(b) we show the action of the associator . As a function of sets, this natural transformation of profunctors has type $\mathcal{C}(A,(B \otimes C) \otimes D) \to \mathcal{C}(A, B \otimes (C \otimes D))$, and acts by post-composition of $(B \otimes C) \otimes D \xrightarrow{\alpha_{B,C,D}} B \otimes (C \otimes D)$.

The definition of profunctor composition for two profunctors $F\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathbf{Set}$ and $G\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{B} \to \mathbf{Set}$ is $G \diamond F :=\int^b G(-,b) \times F(b,=)$, where $\int$ denotes a coend [14, Equation 5.1]. This is interpreted in **Set** as a disjoint union quotiented by the least equivalence relation generated by $(f \cdot g, h) \sim (f, g \cdot h)$. From an internal string diagram perspective, this precisely says that morphisms can 'move freely' through boundary circles. We illustrate this in Figure 1(c).

A formal treatment of internal string diagrams is given in Bartlett et al. [4, § 4], which establishes that it is a sound calculus for reasoning about $\mathcal{M}^{\mathsf{RA}}$.

3 The traced pseudomonoid presentation

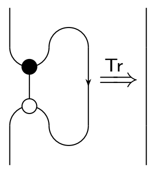
In this section, we introduce our main contribution: the algebraic gadget in **Prof** which admits traced monoidal (Cauchy complete) categories as interpretations. This offers an *external* perspective on traced monoidal categories, akin to how monoidal categories can be viewed externally as pseudomonoids internal to **Cat**, versus a category equipped with *internal* structure.

3.1 Traced monoidal categories

In our framework, we seek to capture the standard notion of traced monoidal category, recalled in Appendix A, in terms of a presentation.

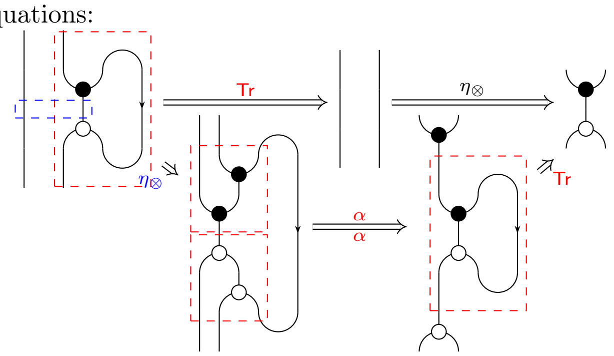
Definition 3.1 The *traced pseudomonoid presentation* $\mathcal{T}$ is given by the data of $\mathcal{M}^{\mathsf{RA}}$, and additionally:

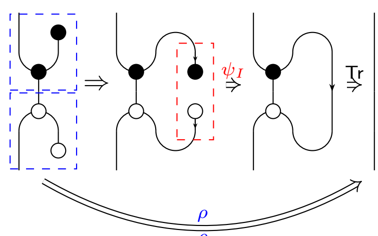
- a generating 2-morphism:



(Tr)

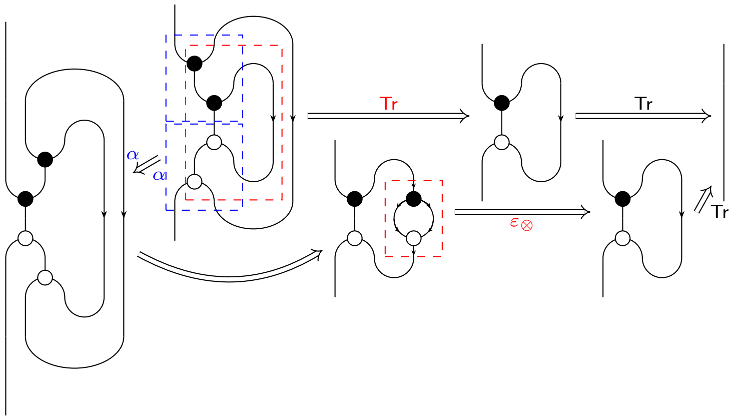
- equations:





(Tr-van-I)

(Tr-sup)



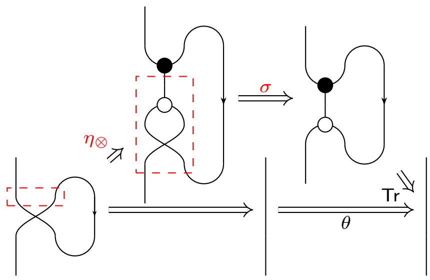
(Tr-van-⊗)

The following theorem records the fact that our presentation $\mathcal{T}$ correctly encodes traced monoidal categories, up to Cauchy completeness. A nice detail of the proof, given in Appendix B, is that the traced monoidal category axioms (tight) and (sli) arise 'for free', simply because the 2-morphism Tr is a natural transformation of profunctors.

Theorem 3.2 *Interpretations of $\mathcal{T}$ in* **Prof** *correspond to Cauchy complete traced monoidal categories.*

An additional consideration is that a balanced traced monoidal category has additional equations it must satisfy, and these are small deviations from straightforwardly replacing $\mathcal{T}$ with its balanced version (the presentation obtained by substituting $\mathcal{L}$ for $\mathcal{M}$ in Definition 3.1).

Definition 3.3 The *balanced traced pseudomonoid presentation* $\mathcal{T}'$ is given by the balanced version of $\mathcal{T}$, and additionally the equation:



($\mathcal{T}'$-yank)

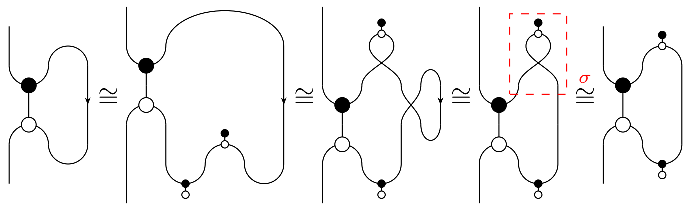
Theorem 3.4 *Interpretations of $\mathcal{T}'$ in* **Prof** *correspond to Cauchy complete balanced traced monoidal categories.*

The symmetric version is a special case of this where θ is given by the identity 2-morphism, as per Remark 2.14.

4 Braided autonomous categories are traced

It is known that every tortile (autonomous + pivotal + balanced) category admits a canonical trace [12, § 3]. This result is known to generalise to any braided autonomous category, which we replicate in our framework. Furthermore, we justify that such a trace is not necessarily unique.

The key observation comes from the fact that in the braided autonomous pseudomonoid presentation, we have a chain of isomorphisms:

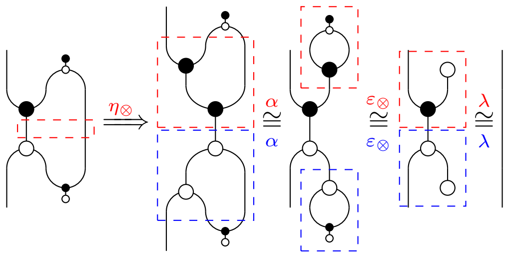


(Tr-NTr)

The first isomorphism utilises the duality generated by $(\curlyvee, \curlywedge)$ in the autonomous pseudomonoid presentation, the second is an isotopy, the third utilises the compact structure of **Prof**, and the fourth utilises braiding.

Secondly, in any monoidal category, we have the following 2-morphism.

Definition 4.1 For a monoidal category, its *nearly tracing* is the following 2-morphism:

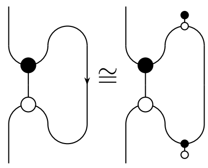


(NTr)

It is then reasonable to come up with analogues to the tracing axioms, and ask if Equation (NTr) satisfies them.

Proposition 4.2 NTr *satisfies the nearly-tracing axioms (Appendix B) for any Cauchy complete monoidal category.*

Corollary 4.3 *Any Cauchy complete category for which the following isomorphism exists, can be equipped with a trace:*



Thanks to Equation (Tr-NTr), this implies the standard result that a braided monoidal category is traced.

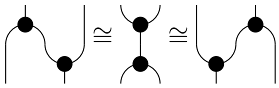
Remark 4.4 This trace is not necessarily unique, because we can arbitrarily apply a twist 2-morphism θ for any available twisting. By fixing a chosen twist (in the sense of a balanced monoidal category), the trace becomes canonical as in Joyal, Street and Verity [12, § 3].

5 On traced $*$-autonomous categories

In this section we apply our traced pseudomonoid technology to study phenomena related to $*$-autonomous categories. Firstly, we show that for a Cauchy complete $*$-autonomous category, a right $\otimes$-trace is equivalent to a left ⅋-trace. Secondly, inspired by the result of Hajgató and Hasegawa [10] that every traced symmetric $*$-autonomous category is compact closed, we seek to investigate the non-symmetric version of this statement. We use our traced pseudomonoid approach to derive a sufficient condition for a traced $*$-autonomous category to be autonomous.

5.1 ∗-Autonomous categories

Definition 5.1 The *Frobenius pseudomonoid presentation* $\mathcal{F}$ is obtained by combining the pseudomonoid presentation (·, ⅄, •) with the pseudocomonoid presentation (·, Y, •) on the same object, and additionally:

- invertible generating 2-morphisms:  (Frob)
- equational structure making these 2-morphisms coherent.

Dunn and Vicary [8, Definition 1.2] give an explicit presentation which they prove to be coherent, with equational structure given by the so-called 'swallowtail equations', which we omit here for brevity, using coherence directly to derive our results. In this case, the coherence result states that, given two parallel 2-morphisms P, Q, whose common source 1-morphism is connected and acyclic as a string diagram, then $P = Q$.

As above, we are interested in the presentation obtained by freely adding right adjoints to particular generating 1-morphisms.

Definition 5.2 The *right-adjoint Frobenius pseudomonoid presentation* $\mathcal{F}^*$ is obtained from $\mathcal{F}$ by adding right-adjoint generating 1-morphisms for the pseudomonoid multiplication and unit: ⅄ ⊣ Y and • ⊣ ∘.

This structure corresponds to Cauchy complete ∗-autonomous categories, which we outline next. Full details can be found in Dunn and Vicary [8, § 2.7].

Theorem 5.3 *Interpretations of* $\mathcal{F}^*$ *in* **Prof** *correspond to Cauchy complete ∗-autonomous categories, where the generating 1-morphisms ⅄ and • represent ⅋ and ⊥ respectively, and the derived 1-morphisms ⅄ and ∘ [8, Definition 2.33] represent ⊗ and I respectively.*

Definition 5.4 The *traced ∗-autonomous presentation* $\mathcal{T}^*$ is obtained by combining $\mathcal{T}$ and $\mathcal{F}^*$, i.e. the presentation containing a right-adjoint Frobenius pseudomonoid as in $\mathcal{F}^*$, where the derived left-adjoint pseudomonoid representing the tensor product (·, ⅄, ∘) is additionally a traced pseudomonoid.

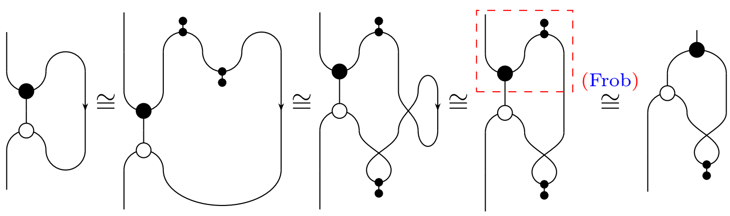
Its constituent parts are interpreted in **Prof** by Cauchy complete traced and ∗-autonomous categories respectively, so the combined presentation is interpreted by a category which is simultaneously traced and ∗-autonomous.

5.2 Rotations

In this section we show that for a ∗-autonomous category, a left ⊗-trace is equivalent to a right ⅋-trace. The idea is that ⊗ and ⅋ are related by duality, and that tracing can be transported through this duality. Furthermore, the dual trace obtained is 'rotated'. Before proving this, we will first try to simplify $\mathcal{T}^*$.

A complication is that $\mathcal{T}^*$ refers to composites containing the 1-morphism ⅄, which with respect to $\mathcal{F}^*$ is a derived 1-morphism, as opposed to a generating 1-morphism. To simplify this, we shall progressively rewrite the data of $\mathcal{T}^*$ in terms of generating 1-morphisms. First, we dispense with the compact closed assumption in the ambient bicategory (**Prof**).

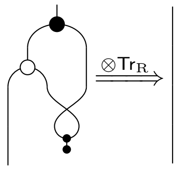
Remark 5.5 We utilise the Frobenius duality generated by (Y, ⅄), similarly to Equation (Tr-NTr), obtaining the isomorphism:



Notice that such a 1-morphism may be interpreted in any monoidal bicategory with duals, as opposed to the stronger requirement of compactness. Henceforth, we will derive a tracing presentation with respect to this 1-morphism.

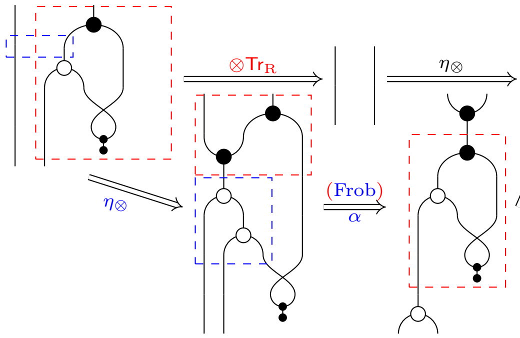
Definition 5.6 The *rotational right $\otimes$-traced $*$-autonomous pseudomonoid presentation* is given by the data of $\mathcal{F}^*$, and additionally:

- a generating 2-morphism:

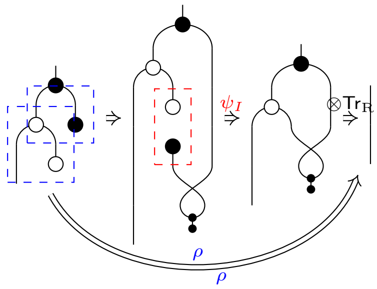


($\otimes\mathsf{Tr}_{\mathrm{R}}$)

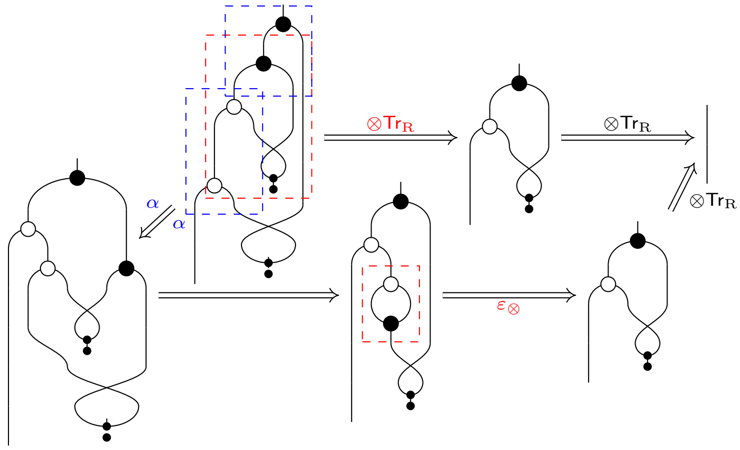
- equations witnessing the axioms of traced monoidal categories:



($\otimes\mathsf{Tr}_{\mathrm{R}}$-sup)



($\otimes\mathsf{Tr}_{\mathrm{R}}$-van-$I$)



($\otimes\mathsf{Tr}_{\mathrm{R}}$-van-$\otimes$)

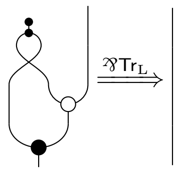
Proposition 5.7 *Interpretations of the rotational right $\otimes$-traced $*$-autonomous pseudomonoid presentation are Cauchy complete traced $*$-autonomous categories, where $\otimes$ is traced on the right.*

Proof From Theorem 3.2, it suffices to show that we can recover all the data of $\mathcal{T}$ from this presentation. This holds by transporting along the isomorphism defined in Remark 5.5. □

We have weakened our setting to a symmetric monoidal bicategory with duals, rather than a compact closed bicategory (notice the lack of cups and caps in our string diagrams). However, they still mention $\curlywedge$, as we would like to discuss a traced monoidal category where the trace is with respect to $\otimes$ (as opposed to the other tensor product $\parr$).

Definition 5.8 The *rotational left $\parr$-traced $*$-autonomous pseudomonoid presentation* is given by the data of $\mathcal{F}^*$, and additionally:

- a generating 2-morphism:



($\parr\mathsf{Tr}_\mathrm{L}$)

- equations witnessing the axioms of traced monoidal categories, analogous to $\otimes\mathsf{Tr}_\mathrm{R}$.

Proposition 5.9 *Interpretations of the rotational left $\parr$-traced $*$-autonomous pseudomonoid presentation are Cauchy complete traced $*$-autonomous categories, where $\parr$ is traced on the left.*

Proof Symmetric to the proof of Theorem 5.7. □

We can now state the main result of this section.

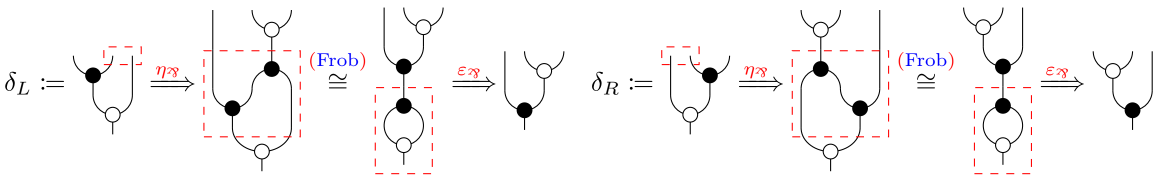
Theorem 5.10 *For a Cauchy complete $*$-autonomous category, a right $\otimes$-trace and a left $\parr$-trace are equivalent.*

5.3 Invertible linear distributivity

Definition 5.11 A $*$-autonomous category has distinguished maps called *linear distributors*; for all objects A, B, and C:

$$A \otimes (B \parr C) \xrightarrow{\delta_L} (A \otimes B) \parr C, \quad (A \parr B) \otimes C \xrightarrow{\delta_R} A \parr (B \otimes C).$$

With respect to $\mathcal{F}^*$, these are corepresented by the composite 2-morphisms:

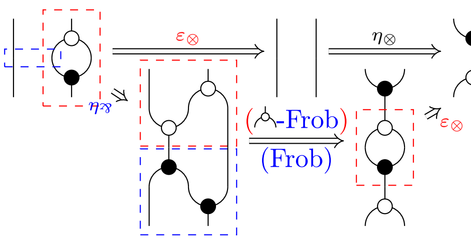


Definition 5.12 The *invertibly linear distributive presentation* $\mathcal{D}$ is obtained by adding inverses to the linear distributor 2-morphisms in $\mathcal{F}^*$.

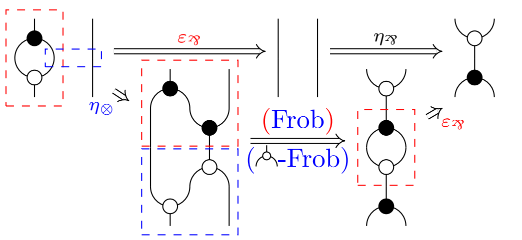
This is simpler to work with, and is equivalent to the data of Definition 2.15 by bending the open leg of with .

Recalling that an autonomous category is precisely a $*$-autonomous category which has invertible linear distributors, we can state the following result, which references eq. (-Frob) given in Appendix C.1. This is a first step towards a non-symmetric version of the result of Hajgató and Hasegawa [10], that every traced symmetric $*$-autonomous category is autonomous.

Proposition 5.13 *Any Cauchy complete left and right $\otimes$-traced $*$-autonomous category for which the following equations, along with their symmetric analogues, hold is autonomous:*



($\varepsilon_\otimes$-sup)



($\varepsilon_\parr$-sup)

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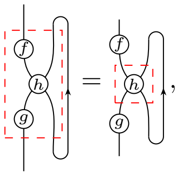
A Traced monoidal categories

We recall the standard definition of traced monoidal category [12].

Definition A.1 A right $\otimes$-traced monoidal category $\mathcal{C}$ is one in which for every morphism $f\colon A\otimes X\to B\otimes X$, there exists a morphism $\mathsf{Tr}^X_{A,B}\colon A\to B$ called its *trace* (along X). Tracing is subject to the following conditions [6]:

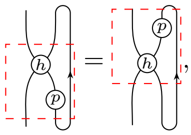
- for every $A\xrightarrow{f}A'$, $A'\otimes X\xrightarrow{h}B\otimes X$, and $B\xrightarrow{g}B'$,

$$\mathsf{Tr}^X_{A,B'}(g\otimes\mathsf{id}_X\circ h\circ f\otimes\mathsf{id}_X)=g\circ\mathsf{Tr}^X_{A',B}(h)\circ f \tag{tight}$$



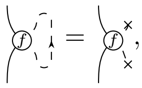
- for every $X\xrightarrow{p}X'$, and $A\otimes X'\xrightarrow{h}B\otimes X$,

$$\mathsf{Tr}^{X'}_{A,B}(\mathsf{id}_B\otimes p\circ h)=\mathsf{Tr}^X_{A,B}(h\circ\mathsf{id}_A\otimes p) \tag{sli}$$



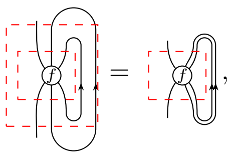
- for every $A\otimes I\xrightarrow{f}B\otimes I$,

$$\mathsf{Tr}^{I}{}_{A,B}(f)=\rho_B\circ f\circ\rho_A^{-1} \tag{van-I}$$



- for every $A\otimes X\otimes Y\xrightarrow{f}B\otimes X\otimes Y$,

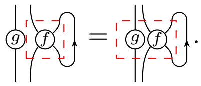
$$\mathsf{Tr}^X_{A,B}(\mathsf{Tr}^Y_{A\otimes X,B\otimes X}(f))=\mathsf{Tr}^{X\otimes Y}_{A,B}(f) \tag{van-$\otimes$}$$



[6] These diagrams go top-down, and we use dashed boxes to indicate which part of the diagram is being traced when ambiguous.

- for every $C \xrightarrow{g} D$ and $A \otimes X \xrightarrow{f} B \otimes X$,

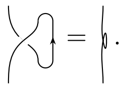
$$g \otimes \mathsf{Tr}^X_{A,B}(f) = \mathsf{Tr}^X_{C\otimes A, D\otimes B}(g \otimes f) \tag{sup}$$



In the case where $\mathcal{C}$ is a balanced monoidal structure, we also additionally require that

- for the braiding natural isomorphism $X \otimes X \xrightarrow{\sigma_{X,X}} X \otimes X$,

$$\mathsf{Tr}^X_{X,X}(\sigma_{X,X}) = \theta_X \tag{YANK}$$



In the case where $\otimes$ is not strictly associative, instead of Equations (van-$\otimes$) and (sup) we have:

(i) for every $(A \otimes X) \otimes Y \xrightarrow{f} (B \otimes X) \otimes Y$,

$$\mathsf{Tr}^X_{X,Y}(\mathsf{Tr}^Y_{A\otimes X, B\otimes X}(f)) = \mathsf{Tr}^{X\otimes Y}_{A,B}(\alpha_{B,X,Y} \circ f \circ \alpha^{-1}_{A,X,Y}), \tag{van-$\otimes$-weak}$$

(ii) for every $C \xrightarrow{g} D$ and $A \otimes X \xrightarrow{f} B \otimes X$,

$$g \otimes \mathsf{Tr}^X_{A,B}(f) = \mathsf{Tr}^X_{C\otimes A, D\otimes B}(\alpha_{D,B,X} \circ g \otimes f \circ \alpha^{-1}_{C,A,X}). \tag{sup-weak}$$

Our diagrams are unchanged, as the associators are absorbed into the geometry. It is also clear that in a strict setting, the associators degenerate to identity morphisms, and this more general formulation coincides with the previous. In general, we will not assume $\otimes$-strictness.

Theorem 3.2. *Interpretations of $\mathcal{T}$ in* **Prof** *correspond to Cauchy complete traced monoidal categories.*

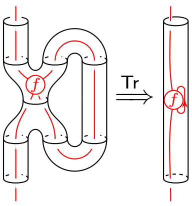
Proof The 2-morphism Tr has the following type:

$$\int^Z \mathcal{C}(- \otimes Z, = \otimes Z) \xRightarrow{\mathsf{Tr}} \mathcal{C}(-, =),$$

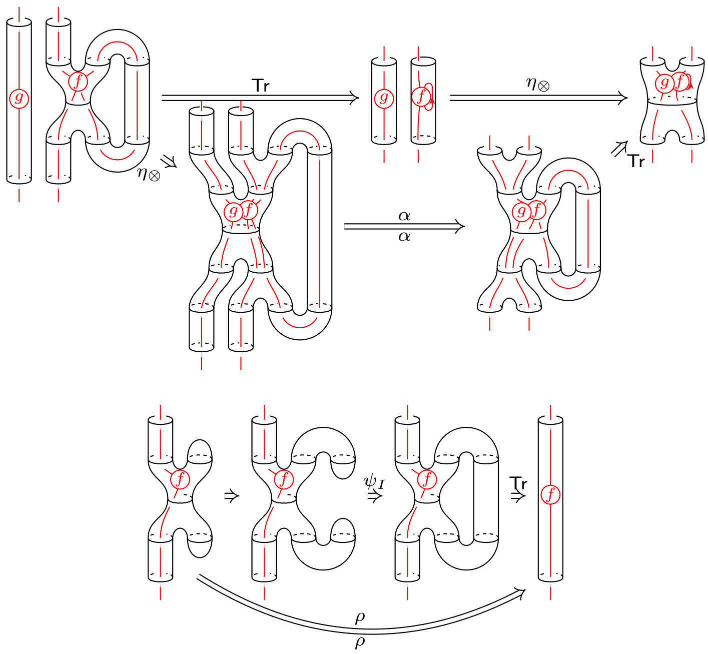
which at component (A, B) we choose to interpret as the map on Hom-sets:

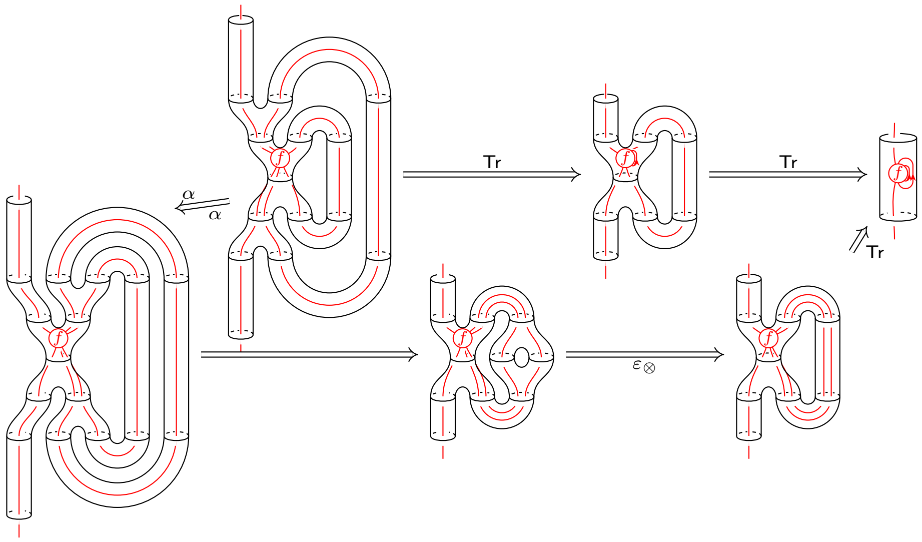
$$\begin{array}{ccccc} \mathcal{C}(A \otimes X, B \otimes X) & \to & \int^Z \mathcal{C}(A \otimes Z, B \otimes Z) & \xrightarrow{\mathsf{Tr}_{A,B}} & \mathcal{C}(A, B) \\ A \otimes X \xrightarrow{f} B \otimes X & \mapsto & [A \otimes X \xrightarrow{f} B \otimes X] & \longmapsto & A \xrightarrow{\mathsf{Tr}^X_{A,B}} B, \end{array}$$

where the first map is given by the universal cowedge of the coend. This uniformly maps a morphism $f \in \mathcal{C}(A \otimes X, B \otimes X)$ to some morphism in $\mathcal{C}(A, B)$, for any object X of $\mathcal{C}$. We suggestively name the resultant morphism $\mathsf{Tr}^X_{A,B}$, alluding that this morphism is the right $\otimes$-trace of f along X according to the standard definition of trace. In accordance with this, we graphically depict this morphism with a backwards loop, just as we did in Equation (1). Thus we describe the action of the Tr 2-morphism on internal strings as:



The sliding and tightening (Equations (sli) and (tight), also called dinaturality and naturality respectively) axioms follow automatically, by virtue of Figure 1(c). Finally, we justify that Equations (Tr-sup) to (Tr-van-⊗) are equivalent to the vanishing and superposing axioms (Equations (van-I) to (sup-weak)):

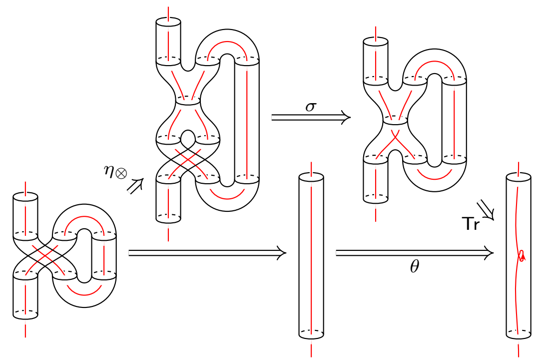




□

Theorem 3.4. *Interpretations of $\mathcal{T}'$ in* **Prof** *correspond to Cauchy complete balanced traced monoidal categories.*

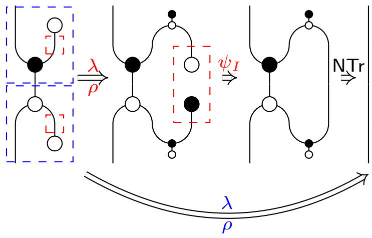
Proof It suffices to show that Equation (YANK) additionally holds:



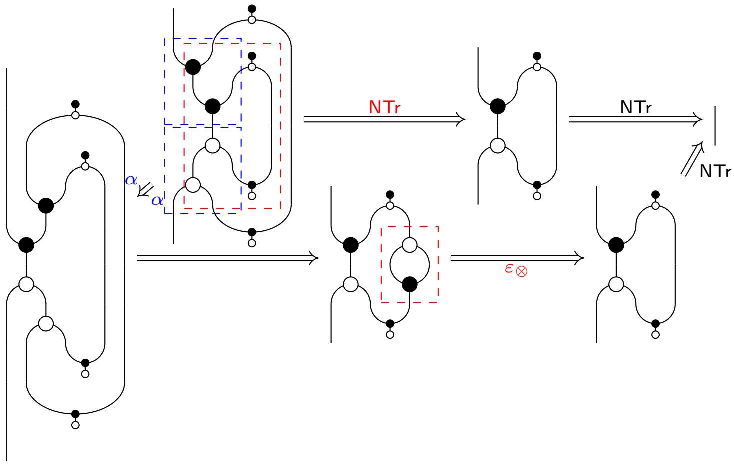
□

B Nearly-tracing axioms

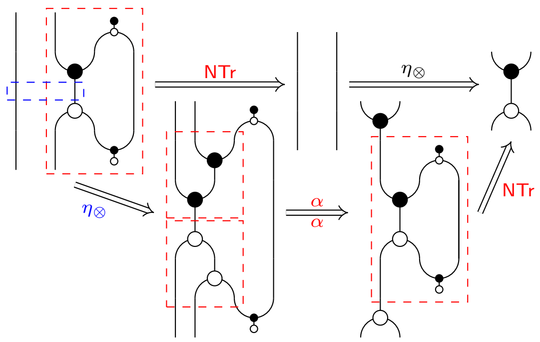
Definition B.1



(NTr-van-I)



(NTr-van-⊗)



(NTr-sup)

C ∗-Autonomous categories

Theorem 5.3. *Interpretations of $\mathcal{F}^*$ in* **Prof** *correspond to Cauchy complete ∗-autonomous categories, where the generating 1-morphisms and represent ⅋ and ⊥ respectively, and the derived 1-morphisms and [8, Definition 2.33] represent ⊗ and I respectively.*

Proof (Sketch) The key idea lies within ∗-autonomous categories as precisely semantics for multiplicative linear logic. A ∗-autonomous category may be regarded as one which carries two distinct but interacting tensor products, which serve as multiplicative conjunction and disjunction [7] :

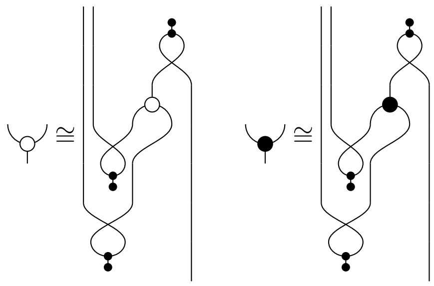
$$A \otimes B, \quad \text{and} \quad A ⅋ B,$$

read 'tensor' and 'par' respectively.

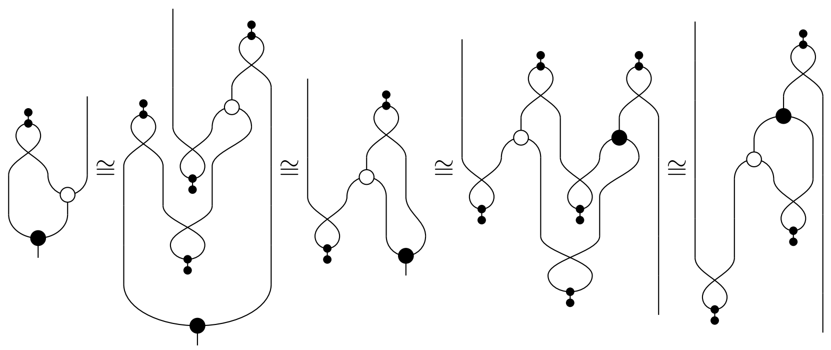
The adjunction ⊣ represents/corepresents ⅋ [8] , and similarly some adjunction involving to represent/corepresent ⊗ is expected. Rather than freely adding an adjoint for the comonoid data, emphasising that we explicitly did not do this in defining $\mathcal{F}^*$, the left adjoint of can be constructed from the existing data in $\mathcal{F}^*$ by transporting the free right-adjoint along the duality induced by the Frobenius structure. All in all, we have free adjunctions ⊣ and ⊣ , along with derived adjunctions ⊣ and ⊣ . □

Theorem 5.10. *For a Cauchy complete ∗-autonomous category, a right ⊗-trace and a left ⅋-trace are equivalent.*

Proof First, observe that we have coherent isomorphisms:



This yields a coherent isomorphism:

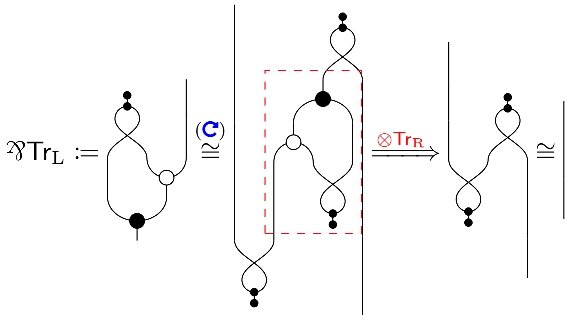


(C)

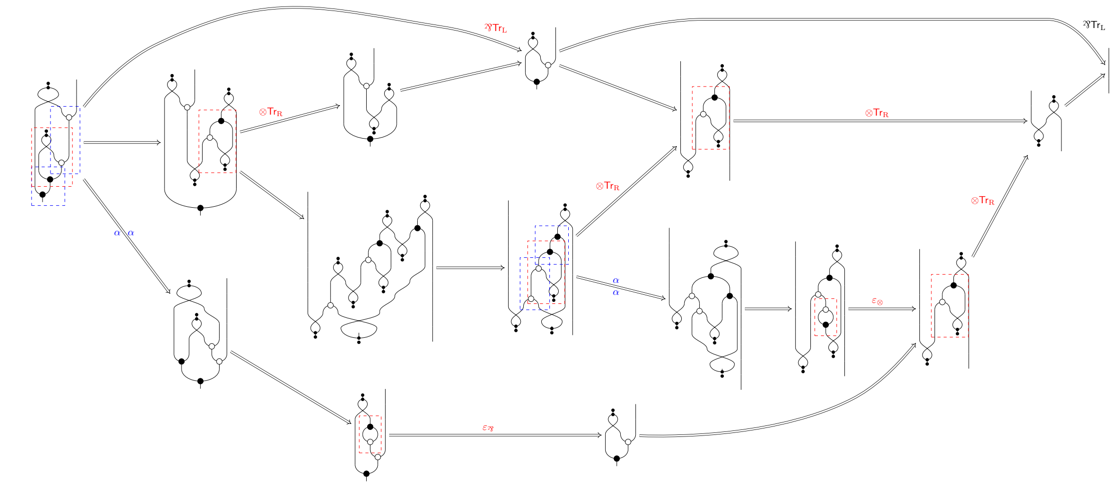
[7] We describe the product operation, but the reasoning is analogous for the unit.

[8] This is why we started with a black monoid, as opposed to previously where we had a white monoid representing ⊗. The reason for this convention is that the black data is coherent, and this colour coding makes it easy to identify those segments.

Now, assuming a right ⊗-tracing structure, define a left ⅋-trace by:



It suffices to show that this derived 2-morphism satisfies the axioms required of a left ⅋-trace; this is accomplished by transporting the assumed right ⊗-tracing axioms across ↻. We demonstrate the case for vanishing along ⅋:

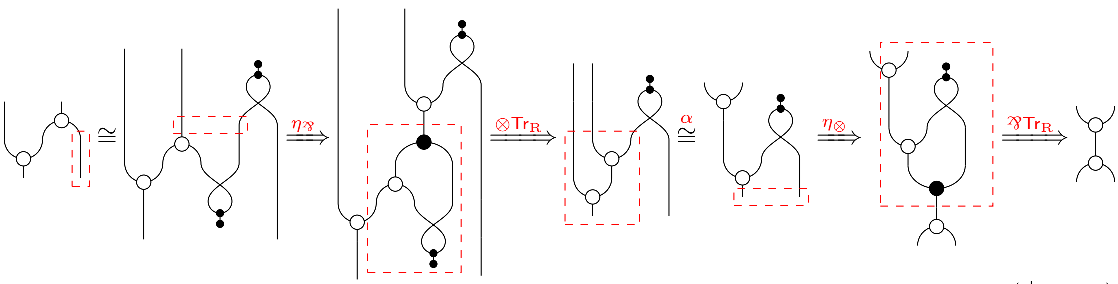


The other axioms are analogous. □

C.1 White Frobeniusator

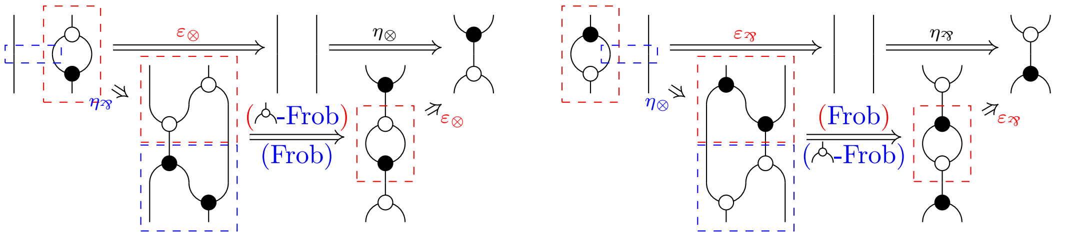
Here we derive a white 'Frobenius' 2-morphism, from $\otimes\mathsf{Tr}_{\mathrm{R}}$ and $⅋\mathsf{Tr}_{\mathrm{R}}$ (equivalently, $\otimes\mathsf{Tr}_{\mathrm{L}}$), and find two equations we would like it to satisfy. For brevity, our aim is to show that δ_R inverts, but for δ_L the symmetric 'Frobenius' 2-morphism and associated conditions are required.

Definition C.1



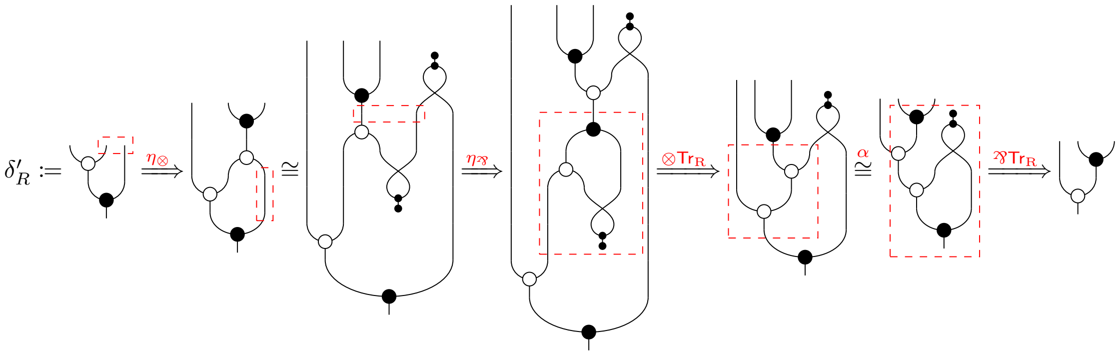
(⅄-Frob)

Proposition 5.13. *Any Cauchy complete left and right $\otimes$-traced $*$-autonomous category for which the following equations, along with their symmetric analogues, hold is autonomous:*



$(\varepsilon_{\otimes}\text{-sup})$ $(\varepsilon_{\parr}\text{-sup})$

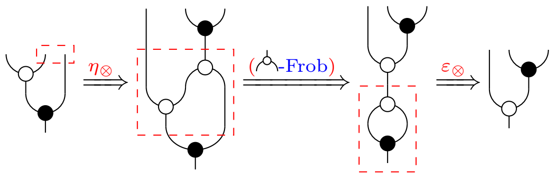
Proof Let δ'_R be defined by:



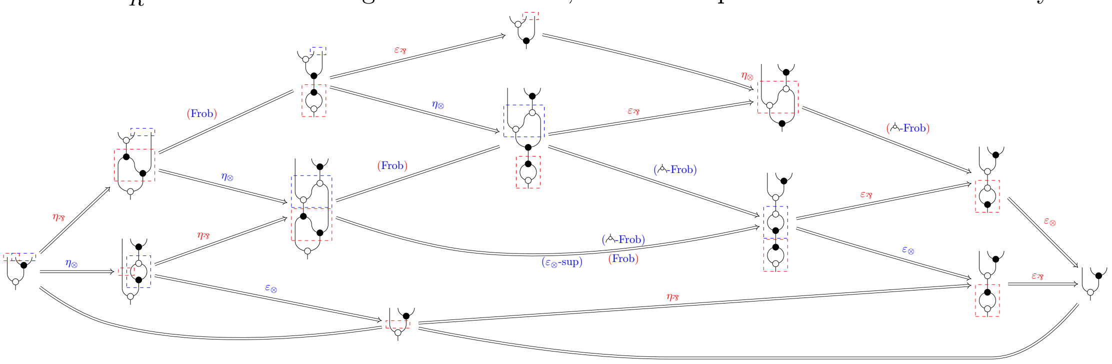
This is an inverse to δ_R, i.e. $\delta_R^{-1} = \delta'_R$. Without loss of generality, we focus on δ_R, as the argument for δ_L is symmetric.

We first define a white 'Frobenius' 2-morphism as in Equation (⅄-Frob).

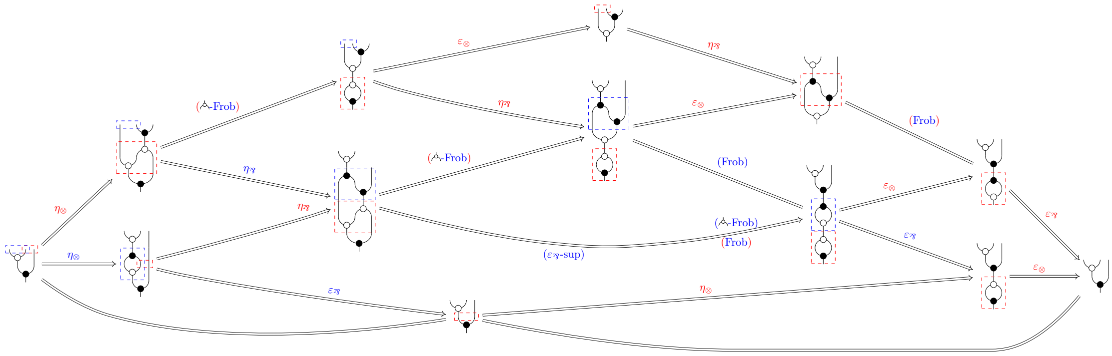
Then δ'_R is a simplification of the 2-morphism:



For δ'_R to be a genuine inverse, we require the commutativity of



i.e. that $\delta'_R \circ \delta_R = \mathsf{id}$, and



i.e. that $\delta_R \circ \delta'_R = \mathsf{id}$. Each tetragon is a naturality square, and each triangle is either a composition of 2-morphisms, or an adjunction triangle equation. The cells which do not immediately commute are the pentagons at the bottom of each diagram, which when isolated yield Equations ($\varepsilon_{⅋}$-sup) and ($\varepsilon_{\otimes}$-sup) as desiderata.

We allude to the similarity of these equations to the tracing superposing axiom, as tracing is similar to the $\varepsilon_{⅋}$ 2-morphism, with a twist.

For the first diagram, for $\delta'_R \circ \delta_R = \mathsf{id}$, we can further simplify to the diagram on the next page. The resulting equation is tantamount to verifying in $\mathcal{C}$ that

$$\mathsf{id}_{A\otimes B} = \mathsf{Tr}_{⅋}^{B^*}\left(\mathsf{coeval}_{B,A} \circ \mathsf{Tr}_{\otimes}^{B}(\mathsf{eval}_{B,A\otimes B})\right),$$

utilising the isomorphism $A \multimapinv B \cong A ⅋ B^*$. □