

# Multinomial and Hypergeometric distributions in Markov Categories

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## Abstract

Markov categories, having tensors with copying and discarding, provide a setting for categorical probability. This paper uses finite colimits and what we call uniform states in such Markov categories to define a (fixed size) multiset functor, with basic operations for sums and zips of multisets, and a graded monad structure. Multisets can be used to represent both urns filled with coloured balls and also draws of multiple balls from such urns. The main contribution of this paper is the abstract definition of multinomial and hypergeometric distributions on multisets, as draws. It is shown that these operations interact appropriately with various operations on multisets.

*Keywords:* Probability theory, category theory, multisets, multinomial distribution, hypergeometric distribution

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## 1 Introduction

Given the current reliance on the probabilistic analysis of huge datasets, it is important to have a good formal understanding of what may be called the logic of probability. In this line of work there is growing interest in the axiomatisation of probability theory, using *e.g.* category theory — also called ‘synthetic’ probability theory. Several efforts and approaches can be distinguished. We list a few of them, without claim to completeness.

- (i) Probabilistic programming languages that incorporate updating (conditioning) and/or higher order features, see *e.g.* [10,11,12,32,34,13].
- (ii) The compositional approach to Bayesian networks [8,16] and to Bayesian reasoning [9,27,25].
- (iii) The use of diagrammatic methods in (quantum) foundations and probability, see [7] for an overview.
- (iv) Study of ‘probability monads’, *e.g.* in [30,23].
- (v) Axiomatisation of disintegration as key probabilistic technique, see *e.g.* [17,3,19,18], and also [5].
- (vi) Exploration of categorical structures, such as compact closed categories [1,33] or effectuses [21,4].

These topics cover both ordinary (classical) probability as well as quantum probability.

An issue that we are particularly interested in is the interplay between multisets (a.k.a. bags) and (probability) distributions (see *e.g.* [26]). Multisets play a fundamental role in probability theory, for instance as representations of urns with coloured balls, and also of draws from such urns. More generally, in learning, collections of data items, possibly occurring multiple times, are properly represented as multisets. Multinomial and hypergeometric distributions assign probabilities to draws from an urn, and can thus be represented as distributions on multisets. Multinomial distributions capture draws *with* replacement, whereas hypergeometric distributions capture draws *without* replacement. In the hypergeometric case the number of balls in the urn decreases with every draw, but in the multinomial case the urn remains unchanged — and can thus be represented as a distribution.

A basic, unsolved question that arises is: should one axiomatise distributions inside the world of multisets (via causal maps, as *e.g.* in [4, Sec. 6] or [6]), or should multisets be described in the world of distributions?

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Briefly: do multisets or distributions come first? The question is highly relevant for axiomatisation, since for instance, in the world of multisets one assumes biproducts  $\oplus$ , whereas in a world with distributions coproducts  $+$  play a leading role. These differences can also be expressed in terms of preservation properties of monads [30]. Of course, there are many more differences, but also similarities, such as presence of a monoidal structure  $\otimes$  for parallel composition.

Most attention so far has gone to the first approach — with multisets first. Recently, the author published a paper [24] that details the distributive law  $\mathcal{M}\mathcal{D} \Rightarrow \mathcal{D}\mathcal{M}$  of multisets over distributions, called the parallel multinomial law *pml*. As a result, multisets  $\mathcal{M}$  can be lifted to the Kleisli category  $\mathcal{Kl}(\mathcal{D})$  of the distribution monad. Actually, what turned out to be most relevant, and well-behaved, is the functor  $\mathcal{M}[K]$  that takes  $K$ -sized multisets only, for a number  $K \in \mathbb{N}$ . Via the lifted functor  $\mathcal{M}[K]: \mathcal{Kl}(\mathcal{D}) \rightarrow \mathcal{Kl}(\mathcal{D})$  multisets appear in the world of distributions — following the second approach, with distributions first.

The paper [24] demonstrated that besides the distributive law, several other probabilistic operations behave well in the paper’s setting, notably multinomial and hypergeometric distributions, and a new form of zipping for multisets, called multizip. The aim of the current paper is to reconstruct many of these results from [24], in an axiomatic setting. Multisets  $\mathcal{M}[K]$  of a fixed size will be defined via a suitable quotient (see [28,2]), and many operations are then derived from the associated universal property, including sums and zips of multisets and multinomial and hypergeometric maps. Any useful axiomatisation of probability theory should include at least such basic distributions. Interestingly, the distributive law *pml* that plays such a central role in [24] is completely absent here. The reason is that *pml* is used for lifting  $\mathcal{M}[K]$  to  $\mathcal{Kl}(\mathcal{D})$  in [24], whereas here our aim is to axiomatise  $\mathcal{M}[K]$  on categories like  $\mathcal{Kl}(\mathcal{D})$ . The existence of *pml* justifies what we do here.

The current paper can be read without knowing about [24], but familiarity with that paper does help to understand why certain choices are made here. Our axiomatisation happens in a monoidal category with copying and discarding — like in [3], called *Markov category* in [17]. Here we shall use Markov categories with finite colimits, plus distributivity of  $\otimes$  over  $+$ , and what we call ‘uniform states’. Due to space constraints we focus solely on the axiomatisation itself, and not on categories that possibly satisfy these requirements. Our leading examples are the Kleisli categories  $\mathcal{Kl}(\mathcal{D})$  and  $\mathcal{Kl}(\mathcal{G})$  of the distribution and Giry monads  $\mathcal{D}$  and  $\mathcal{G}$ , for discrete and continuous probability. We refer to [17] (and also [14]) for further instances of Markov categories.

The line of axiomatisation proposed here is a first step, with several loose ends, and is far from completed. Still this direction is already of interest in this early stage, because it leads to representation of practically relevant distributions. Our approach has a clear discrete focus so far, centered around multinomial and hypergeometric distributions, even though it applies in categories for continuous probability too. But it does not cover typical continuous distributions like normal, beta, gamma, *etc.*, for which the approach of [13] could be useful. Our axiomatisation is based on multisets, and includes sums and multizips of such multisets, but not tensors of multisets. Although tensors of multisets are a basic operation, they do not seem to fit in the current set up, because they are not natural w.r.t. Kleisli maps, see the discussion at the end of this paper.

This article is organised as follows. It first introduces Markov categories with colimits and uses them to define multisets in Sections 2 and 3. The additional probabilistic requirements, in the form of uniform states are defined in Section 4. Sections 5, 6 and 7 introduce basic operations on multisets, such as arrangement and frequentist learning, draw-and-delete, and sums and zips. Multinomial and hypergeometric distributions are then defined in Section 8 and basic properties are proven, such as proper interaction with frequentist learning, with draw-and-delete, and with multiset zipping.

## 2 Markov categories with colimits

This section briefly introduces the setting in which we will be working. A *Markov category* is a symmetric monoidal category in which:

- the tensor unit is a final object  $1$ , so that tensors  $\otimes$  have projections  $\pi_1: X \otimes Y \rightarrow X \otimes 1 \cong X$ ; this makes the setting affine, see [22,23];
- each object  $X$  carries a comonoid structure  $\delta: X \rightarrow X \otimes X$  for copying, which is commutative and associative and satisfies  $\pi_1 \circ \delta = \text{id}$ , and interacts appropriately with the monoidal structure  $(\otimes, 1)$ . These copiers can be combined to  $n$ -ary form  $\delta[K]: X \rightarrow X^K = X \otimes \cdots \otimes X$ .

We should emphasise that these copiers  $\delta$  are *not* natural. In fact, a map  $f$  may be called *deterministic* if it does commute with copying:  $(f \otimes f) \circ \delta = \delta \circ f$ . It is required that all monoidal isomorphisms are deterministic.

We shall assume that our Markov category has finite colimits, with several additional requirements.

- The coprojections (colimit injections)  $\kappa_i: X_i \rightarrow \text{colim}_i X_i$  are deterministic, and also the mediating map induced by deterministic maps is itself deterministic;
- Tensors  $\otimes$  distribute over coproducts  $+$ . It makes sense to require that tensors distribute over all finite

colimits — so also over coequalisers — but we need that requirement at one point only, see Theorem 7.3, and so we explicitly require it there.

We shall need one more requirement, namely ‘uniform states’, which will be introduced in Section 4. We shall think of maps of the form  $1 \rightarrow X$  as *distributions*, also called *states*, over  $X$ . More generally, maps  $Y \rightarrow X$  are then  $Y$ -indexed distributions, which can be understood as conditional probabilities  $p(X | Y)$ .

In the remainder of this article we shall work in a fixed Markov category  $\mathbf{C}$  with finite colimits as described above.

### 3 Multisets

The formalisation of multiset in our Markov category  $\mathbf{C}$  is a key, first step in our axiomatisation. We shall focus on multisets of a fixed size  $K$ , that is, on multisets with  $K$  elements in total, including multiplicities. Since multisets can be understood as sequences where the order does not matter, it makes sense to describe multisets  $\mathcal{M}[K](X)$  over  $X$  of size  $K$  as quotient  $X^K \twoheadrightarrow \mathcal{M}[K](X)$  of the object  $X^K = X \otimes \cdots \otimes X$  of sequences of length  $K$ , see also [28,2]. This section only contains the definition and functoriality. The sum and zip of multisets are introduced later on, once we have seen uniform states.

We write, as usual,  $S_K$  for the symmetric group of permutations  $\{1, \dots, K\} \cong \{1, \dots, K\}$ . Each permutation  $\sigma \in S_K$  translates into a (deterministic) isomorphism  $\underline{\sigma}: X^K \cong X^K$  via the monoidal isomorphisms.

**Definition 3.1** For each number  $K \in \mathbb{N}$  and  $X \in \mathbf{C}$  write  $\text{acc}[K]$  for the coequaliser of all (interpreted) permutation maps  $\underline{\sigma}: X^K \cong X^K$ , for  $\sigma \in S_K$ , in:

$$X^K \begin{array}{c} \curvearrowright \\ \vdots \\ \underline{\sigma} \\ \curvearrowleft \end{array} X^K \xrightarrow{\text{acc}[K]} \twoheadrightarrow \mathcal{M}[K](X)$$

We call  $\mathcal{M}[K](X)$  the object of  $K$ -sized multisets on  $X$ . The map  $\text{acc}$  is called accumulator; it turns a list into a multiset by ignoring orderings. We omit the number  $K$  in  $\text{acc}[K]$  when it is clear from the context.

Concretely, in a set-theoretic setting one has:  $\text{acc}[5](a, b, a, b, b) = 2|a| + 3|b| \in \mathcal{M}[5](\{a, b\})$ . We use a ket notation  $| - \rangle$  for multisets, see [24] for more (set-theoretic) details.

We collect some basic facts.

**Lemma 3.2** Consider the accumulator map  $\text{acc}: X^K \twoheadrightarrow \mathcal{M}[K](X)$  from Definition 3.1.

- (i) It is deterministic.
- (ii) It satisfies  $\text{acc} \circ \underline{\sigma} = \text{acc}$ , for each permutation  $\sigma \in S_K$ .
- (iii) It is a natural transformation  $(-)^K \Rightarrow \mathcal{M}[K]$ , when  $\mathcal{M}[K]$  is extended to a functor via:

$$\begin{array}{ccc} X^K & \begin{array}{c} \curvearrowright \\ \vdots \\ \end{array} & X^K \xrightarrow{\text{acc}[K]_X} \twoheadrightarrow \mathcal{M}[K](X) \\ \downarrow f^K & & \downarrow f^K \\ Y^K & \begin{array}{c} \curvearrowright \\ \vdots \\ \end{array} & Y^K \xrightarrow{\text{acc}[K]_Y} \twoheadrightarrow \mathcal{M}[K](Y) \end{array} \quad \text{for } f: X \rightarrow Y.$$

- (iv) Precomposition with copying gives a  $K$ -fold unit map  $\text{acc} \circ \delta[K]: X \rightarrow X^K \rightarrow \mathcal{M}[K](X)$ , which is not natural in  $X$ .

**Proof.** (i) The accumulator map  $\text{acc}$  is deterministic, as coequaliser of deterministic maps, see Section 2.  
(ii) Since  $\text{acc}$  is by construction the coequaliser, we have  $\text{acc} \circ \underline{\sigma} = \text{acc} \circ \underline{\tau}$  for all permutations  $\sigma, \tau \in S_K$ . This holds in particular when we choose  $\tau$  to be the identity permutation.  
(iii) This works since  $f^K \circ \underline{\sigma} = \underline{\sigma} \circ f^K$  for each permutation  $\sigma$ .  
(iv) Naturality fails, since only deterministic, not arbitrary, maps commute with copier  $\delta$ . □

## 4 Uniform states

Let  $\mathbf{C}$  be a Markov category as in Section 2. For each  $n \in \mathbb{N}$  there is an *interpreted number*  $\underline{n} \in \mathbf{C}$ , namely:

$$\underline{n} := \underbrace{1 + \cdots + 1}_{n \text{ times}} \quad \text{where } 1 \in \mathbf{C} \text{ is the final object (tensor unit).}$$

We shall use the sums  $+$  in this definition of  $\underline{n}$  up to isomorphisms. Clearly,  $\underline{1} = 1$  and  $\underline{0} = 0$ , as empty sum. Further,  $\underline{n} + \underline{m} \cong \underline{n+m}$ . The codiagonal map  $\nabla_n = [\text{id}, \dots, \text{id}]: \underline{n} \rightarrow 1$  is the unique map to 1.

By distributivity of  $\otimes$  over  $+$  we get a natural isomorphism  $\underline{n} \otimes X \cong X + \cdots + X$  ( $n$  times). A map of the form  $r: 1 \rightarrow \underline{n}$  will be called a *convex series*, of length  $n$ . Given another such series  $s: 1 \rightarrow \underline{m}$  we write:

$$r \bullet s := \left( 1 \xrightarrow{r} \underline{n} = 1 + \cdots + 1 \xrightarrow{s+\cdots+s} \underline{m} + \cdots + \underline{m} \right).$$

Then, up-to-isomorphism,  $r \bullet s = s \bullet r$ . This follows from a Kelly-Laplaza style argument [29]:

The following definition is typical for a probabilistic setting.

**Definition 4.1** We say that the category  $\mathbf{C}$  has *uniform states* if for each  $n \geq 1$  there is a *uniform state*  $\underline{\perp}_n: 1 \rightarrow \underline{n}$ . These states are required to satisfy:

- (i)  $\sigma \circ \underline{\perp}_n = \underline{\perp}_n$ , for each (interpreted) permutation  $\sigma: \underline{n} \xrightarrow{\cong} \underline{n}$ , of size  $n$ , that is for  $\sigma \in S_n$ ;
- (ii)  $\underline{\perp}_n \otimes \underline{\perp}_m = \underline{\perp}_{n \cdot m}$ , up-to-isomorphism.

We think of  $\underline{\perp}_n: 1 \rightarrow \underline{n}$  as the  $n$  probabilities  $(\frac{1}{n}, \dots, \frac{1}{n})$  adding up to 1, and thus forming a convex series. We can use them to form other convex series, such as:

$$\left( \frac{1}{6}, \frac{1}{2}, \frac{1}{3} \right) := \left( 1 \xrightarrow{\underline{\perp}_6} \underline{6} = 1 + \underline{3} + \underline{2} \xrightarrow{\text{id}_1 + \nabla_3 + \nabla_2} 1 + 1 + 1 = \underline{3} \right).$$

In this way we can form each ‘fractional’ convex series  $(\frac{n_1}{n}, \dots, \frac{n_k}{n})$  as map  $1 \rightarrow \underline{k}$  with  $n = \sum_i n_i$ .

Given a convex series  $r: 1 \rightarrow \underline{n}$ , and an  $n$ -tuple of maps  $f_i: X \rightarrow Y$  we can form the *convex sum*  $\sum_i r \cdot f_i: X \rightarrow Y$  via distributivity of  $\otimes$  over  $+$ .

$$\sum_i r \cdot f_i := \left( X \cong X \otimes 1 \xrightarrow{\text{id} \otimes r} X \otimes \underline{n} \cong X + \cdots + X \xrightarrow{[f_1, \dots, f_n]} Y \right).$$

**Lemma 4.2** Consider convex series  $r, s$  with suitably typed maps.

- (i) Convex sums are preserved by sequential composition:

$$\left( \sum_i r \cdot f_i \right) \circ g = \sum_i r \cdot (f_i \circ g) \quad h \circ \left( \sum_i r \cdot f_i \right) = \sum_i r \cdot (h \circ f_i).$$

- (ii) Convex sums are preserved by parallel composition:

$$\sum_i r \cdot (f_i \otimes g) = \left( \sum_i r \cdot f_i \right) \otimes g \quad \sum_i r \cdot (h \otimes f_i) = h \otimes \left( \sum_i r \cdot f_i \right).$$

(iii) *Convex sums of constant collections are constant:*

$$\sum_i r \cdot f = f \quad \text{and in particular} \quad \sum_i r \cdot \text{id} = \text{id}.$$

(iv)  $(\sum_j s \cdot g_j) \circ (\sum_i r \cdot f_i) = \sum_{j,i} (s \bullet r) \cdot (g_j \circ f_i)$ . □

## 5 Arrangement and frequentist learning

In this section we combine multisets with convex sums to obtain arrangement and frequentist learning operations  $\text{arr}: \mathcal{M}[K](X) \rightarrow X^K$  and  $\text{Flrn}: \mathcal{M}[K](X) \rightarrow X$ . Intuitively, the arrangement map  $\text{arr}$  sends a multiset to the uniform distribution of all sequences that accumulate to the multiset. And the frequentist learning map  $\text{Flrn}$  normalises a multiset into a distribution.

Since the symmetric group  $S_K$  of permutations of a set with  $K$  elements has  $K!$  elements we can define:

$$\text{perm} := \sum_{\sigma \in S_K} \frac{1}{K!} \cdot \sigma : X^K \longrightarrow X^K. \quad (1)$$

The following facts follow readily from Lemma 4.2.

- Lemma 5.1** (i)  $\text{perm}$  is natural in  $X$ ;  
(ii)  $\text{perm} \circ \delta[K] = \delta[K]$ ;  
(iii)  $\text{acc} \circ \text{perm} = \text{acc}$ , for the accumulation map of Definition 3.1.

**Proof.** The first two points follow from Lemma 4.2 in:

$$\text{perm} \circ f^K = \sum_{\sigma \in S_K} \frac{1}{K!} \cdot (\sigma \circ f^K) = \sum_{\sigma \in S_K} \frac{1}{K!} \cdot (f^K \circ \sigma) = f^K \circ \text{perm}$$

$$\text{perm} \circ \delta[K] = \sum_{\sigma \in S_K} \frac{1}{K!} \cdot (\sigma \circ \delta[K]) = \sum_{\sigma \in S_K} \frac{1}{K!} \cdot \delta[K] = \delta[K].$$

For the last point we use Lemma 3.2 (ii).

$$\text{acc} \circ \text{perm} = \text{acc} \circ \left( \sum_{\sigma} \frac{1}{K!} \cdot \sigma \right) = \sum_{\sigma} \frac{1}{K!} \cdot (\text{acc} \circ \sigma) = \sum_{\sigma} \frac{1}{K!} \cdot \text{acc} = \text{acc}. \quad \square$$

For an object  $X \in \mathbf{C}$  and a number  $K \in \mathbb{N}$  there are projections  $\pi_i: X^K \rightarrow X$ , for  $1 \leq i \leq K$ . Next, write, for  $K \geq 1$ ,

$$\varepsilon[K] := \sum_i \frac{1}{K} \cdot \pi_i : X^K \longrightarrow X. \quad (2)$$

**Lemma 5.2** *The map  $\varepsilon[K]$  in (2),*

- (i) *is natural in  $X$ ;*  
(ii) *is the identity  $X \rightarrow X$  for  $K = 1$ ;*  
(iii) *satisfies  $\varepsilon[K] \circ \delta[K] = \text{id}$ , for the  $K$ -fold copier  $\delta[K]: X \rightarrow X^K$ .* □

In a next step we define two basic operations associated with multisets, namely frequentist learning  $\text{Flrn}$  and arrangement  $\text{arr}$ .

**Definition 5.3** For  $X \in \mathbf{C}$ , the universal property of the coequaliser yields frequentist learning maps  $\text{Flrn}: \mathcal{M}[K](X) \rightarrow X$ , when  $K \geq 1$ , and arrangement maps  $\text{arr}: \mathcal{M}[K](X) \rightarrow X^K$ , for all  $K \geq 0$ , in situations:

$$\begin{array}{ccc} X^K \begin{array}{c} \curvearrowright \\ \vdots \\ \curvearrowleft \end{array} X^K & \xrightarrow{\text{acc}} \mathcal{M}[K](X) & X^K \begin{array}{c} \curvearrowright \\ \vdots \\ \curvearrowleft \end{array} X^K \\ & \searrow \varepsilon[K] & \searrow \text{perm} \\ & X & X^K \\ & \downarrow \text{Flrn} & \downarrow \text{arr} \end{array}$$

These definitions work since for each permutation  $\tau \in S_K$  one has:

$$\varepsilon \circ \underline{\tau} = \sum_i \underline{\perp}_K \cdot (\pi_i \circ \underline{\tau}) = \sum_i \underline{\perp}_K \cdot \pi_i = \varepsilon \quad \text{perm} \circ \underline{\tau} = \sum_\sigma \underline{\perp}_{K!} \cdot (\underline{\sigma} \circ \underline{\tau}) = \sum_\sigma \underline{\perp}_{K!} \cdot \underline{\sigma} = \text{perm}.$$

**Lemma 5.4** *In the above situation,*

- (i)  $Flrn$  is a natural transformation  $\mathcal{M}[K] \Rightarrow \text{id}$ ;
- (ii)  $arr$  is a natural transformation  $\mathcal{M}[K] \Rightarrow (-)^K$ ;
- (iii)  $acc \circ arr = \text{id}$
- (iv)  $\underline{\sigma} \circ arr = arr$ , for each  $\sigma \in S_K$ .

**Proof.** (i) For  $f: X \rightarrow Y$  we have  $Flrn \circ \mathcal{M}[K](f) = f \circ Flrn$ , since  $acc$  is epic, using Lemma 5.2 (i):

$$Flrn \circ \mathcal{M}[K](f) \circ acc = Flrn \circ acc \circ f^K = \varepsilon \circ f^K = f \circ \varepsilon = f \circ Flrn \circ acc.$$

(ii) Similarly we are done by:

$$arr \circ \mathcal{M}[K](f) \circ acc = arr \circ acc \circ f^K = perm \circ f^K = f^K \circ perm = f^K \circ arr \circ acc.$$

(iii) The equation  $acc \circ arr = \text{id}$  follows from Lemma 5.1 (iii):

$$acc \circ arr \circ acc = acc \circ perm = acc = \text{id} \circ acc.$$

(iv) Let  $\sigma \in S_K$  be given. We get  $\underline{\sigma} \circ arr = arr$  from:

$$\underline{\sigma} \circ arr \circ acc = \underline{\sigma} \circ perm = \sum_{\tau \in S_K} \underline{\perp}_{K!} \cdot (\underline{\sigma} \circ \underline{\tau}) = \sum_{\tau \in S_K} \underline{\perp}_{K!} \cdot \underline{\tau} = perm = arr \circ acc. \quad \square$$

The following points are expected but useful to make explicit.

- Lemma 5.5** (i)  $\mathcal{M}[0](X)$  is final, so  $\mathcal{M}[0](X) \cong 1$ ;  
(ii)  $acc[1]: X \rightarrow \mathcal{M}[1](X)$  is an isomorphism, with  $arr[1]$  as inverse;  
(iii)  $\mathcal{M}[K](1)$  is also final;  
(iv)  $\mathcal{M}[K](0)$  is final for  $K = 0$  and initial for  $K > 0$ .

- Proof.** (i) Since  $X^0 = 1$  by definition, we get  $acc[0]: 1 \rightarrow \mathcal{M}[0](X)$ , obviously satisfying  $! \circ acc[0] = \text{id}: 1 \rightarrow 1$ . But then also  $acc[0] \circ ! = \text{id}$  since  $acc$  is epic and  $acc[0] \circ ! \circ acc[0] = acc[0]$ .  
(ii) We have  $Flrn \circ acc[1] = \varepsilon[1] = \text{id}$  by Lemma 5.2 (ii). But then also  $acc[1] \circ Flrn = \text{id}$  because  $acc[1]$  is epic. Since also  $acc[1] \circ arr[1] = \text{id}$  we get  $acc[1]^{-1} = arr[1] = Flrn: \mathcal{M}[1](X) \rightarrow X$ , by Lemma 5.4 (iii).  
(iii) We already know that  $\mathcal{M}[K](1)$  is final for  $K = 0$ , by the first point. For  $K > 0$  we can use frequentist learning and use that  $1^K = 1$ , so  $Flrn \circ acc[K] = \text{id}: 1 \rightarrow 1$ . But then also  $acc[K] \circ Flrn = \text{id}$  since  $acc[K]$  is epic.  
(iv) Note that  $0^K = 1$  for  $K = 0$  and  $0^K = 0$  for  $K > 0$ . That  $\mathcal{M}[0](0)$  is final follows from the first point. For  $K > 0$  frequentist learning  $Flrn$  is defined, giving  $Flrn \circ acc[K] = \text{id}: 0 \rightarrow 0$ . But then also  $acc[K] \circ Flrn = \text{id}$  since  $acc[K]$  is epic.  $\square$

Proofs of the next results are relegated to the appendix. The notation  $\binom{n}{K} = \binom{n+K-1}{K}$  is the *multichoose* coefficient. It describes the number of multisets of size  $K$  over an  $n$ -element set, see e.g. [15, II (5.2)]. The same result can be obtained in our abstract setting, in point (ii) below.

**Proposition 5.6** (i) For  $K \geq 0$ , and objects  $X, Y$ ,

$$\mathcal{M}[K](X + Y) \cong \bigoplus_{0 \leq i \leq K} \mathcal{M}[i](X) \otimes \mathcal{M}[K-i](Y).$$

(ii) For a number  $n \geq 1$ ,

$$\mathcal{M}[K](\underline{n}) \cong \underline{\binom{n}{K}} = \binom{n+K-1}{K} \cdot 1.$$

## 6 Uniform deletion

When we think of a multiset in  $\mathcal{M}[K](X)$  as an urn filled with  $K$ -many balls with colours in  $X$ , we would like to have an operation for randomly drawing a (single) ball from the urn. We shall describe this as an operation  $DD: \mathcal{M}[K+1](X) \rightarrow \mathcal{M}[K](X)$ , which we call draw-and-delete.

We fix  $K \in \mathbb{N}$  and  $X \in \mathbf{C}$ . For  $1 \leq i \leq K+1$  we first define maps that remove the  $i$ -th element, and then a uniform deletion map:

$$\widehat{\pi}_i := \underbrace{\text{id} \otimes \cdots \otimes \text{id}}_{i-1 \text{ times}} \otimes ! \otimes \underbrace{\text{id} \otimes \cdots \otimes \text{id}}_{K+1-i \text{ times}} : X^{K+1} \rightarrow X^K \quad \text{and} \quad \text{del}[K] := \sum_{1 \leq i \leq K+1} \underline{\perp}_{K+1} \cdot \widehat{\pi}_i : X^{K+1} \rightarrow X^K.$$

In this definition of  $\widehat{\pi}_i$  we write  $!$  for the map to the final object  $1$ .

**Lemma 6.1** *In the above situation,*

- (i) *the maps  $\widehat{\pi}_i$  and  $\text{del}$  are natural;*
- (ii) *deletion commutes with permutation and with  $\varepsilon$ , as in:*

$$\begin{array}{ccc} X^{K+1} & \xrightarrow{\text{perm}[K+1]} & X^{K+1} \\ \text{del} \downarrow & & \downarrow \text{del} \\ X^K & \xrightarrow{\text{perm}[K]} & X^K \end{array} \qquad \begin{array}{ccc} X^K & \xleftarrow{\text{del}} & X^{K+1} \\ \varepsilon[K] \searrow & & \swarrow \varepsilon[K+1] \\ & X & \end{array}$$

(iii)  $\text{del} \circ \delta[K+1] = \delta[K]$ ;

(iv)  $\text{del} \circ \text{perm}[K+1] = \pi \circ \text{perm}[K+1]$ , for the projection  $\pi = \text{id} \otimes !: X^{K+1} \rightarrow X^K$ , and then also  $\text{del} \circ \text{arr}[K+1] = \pi \circ \text{arr}[K+1]$ .

**Proof.** The first point is obvious, but the other ones requires more care. We use that for each permutation  $\sigma \in S_{K+1}$  and index  $1 \leq i \leq K+1$  there is a permutation  $\tau \in S_K$  and index  $j$  with  $\widehat{\pi}_i \circ \underline{\sigma} = \underline{\tau} \circ \widehat{\pi}_j$ . This yields  $K+1$  times the same  $\tau$ . Hence:

$$\begin{aligned} \text{del} \circ \text{perm}[K+1] &= \left( \sum_{1 \leq i \leq K+1} \underline{\perp}_{K+1} \cdot \widehat{\pi}_i \right) \circ \left( \sum_{\sigma \in S_{K+1}} \underline{\perp}_{(K+1)!} \cdot \underline{\sigma} \right) \\ &= \sum_{1 \leq i \leq K+1} \sum_{\sigma \in S_{K+1}} (\underline{\perp}_{K+1} \otimes \underline{\perp}_{(K+1)!}) \cdot (\widehat{\pi}_i \otimes \underline{\sigma}) \\ &= \sum_{1 \leq i \leq K+1} \sum_{\tau \in S_K} (\underline{\perp}_{K+1} \otimes \underline{\perp}_{K!} \otimes \underline{\perp}_{K+1}) \cdot (\widehat{\pi}_i \otimes \underline{\tau} \otimes \text{id}) \\ &= \sum_{1 \leq i \leq K+1} \sum_{\tau \in S_K} (\underline{\perp}_{K+1} \otimes \underline{\perp}_{K!}) \cdot (\widehat{\pi}_i \otimes \underline{\tau}) = \text{perm}[K] \circ \text{del}. \end{aligned}$$

Similarly, all composites  $\pi_j \circ \widehat{\pi}_i$  consist of  $K$  times the projection  $\pi_i: X^{K+1} \rightarrow X$ . Hence:

$$\begin{aligned} \varepsilon[K] \circ \text{del} &= \left( \sum_{1 \leq j \leq K} \underline{\perp}_K \cdot \pi_j \right) \circ \left( \sum_{1 \leq i \leq K+1} \underline{\perp}_{K+1} \cdot \widehat{\pi}_i \right) \\ &= \sum_{1 \leq j \leq K} \sum_{1 \leq i \leq K+1} (\underline{\perp}_K \otimes \underline{\perp}_{K+1}) \cdot (\text{id} \otimes \pi_i) = \sum_{1 \leq i \leq K+1} \underline{\perp}_{K+1} \cdot \pi_i = \varepsilon[K+1]. \end{aligned}$$

Along the same lines we obtain point (iv).

$$\pi \circ \text{perm}[K+1] = \sum_{\sigma \in S_{K+1}} \underline{\perp}_{(K+1)!} \cdot (\pi \circ \sigma) = \sum_{\tau \in S_K} \sum_{1 \leq i \leq K+1} (\underline{\perp}_{K!} \otimes \underline{\perp}_{K+1}) \cdot (\tau \otimes \pi_i) = \text{perm}[K] \circ \text{del}.$$

Finally, for point (iii) we use:

$$\text{del} \circ \delta[K+1] = \sum_{1 \leq i \leq K+1} \underline{\perp}_{K+1} \cdot (\widehat{\pi}_i \circ \delta[K+1]) = \sum_{1 \leq i \leq K+1} \underline{\perp}_{K+1} \cdot \delta[K] = \delta[K]. \quad \square$$

These results allow us to define a draw-and-delete map  $DD: \mathcal{M}[K+1](X) \rightarrow \mathcal{M}[K](X)$  in:

$$\begin{array}{ccc} X^{K+1} & \begin{array}{c} \curvearrowright \\ \vdots \\ \curvearrowleft \end{array} & X^{K+1} \xrightarrow{\text{acc}} \mathcal{M}[K+1](X) \\ & \searrow \text{del} & \downarrow DD \\ & & X^K \xrightarrow{\text{acc}} \mathcal{M}[K](X) \end{array} \quad (3)$$

**Proposition 6.2** Consider the draw-and-delete map  $DD$  defined in (3). Frequentist learning after draw-and-delete is frequentist learning, as described on the left below.

$$\begin{array}{ccc} \mathcal{M}[K](X) & \xleftarrow{DD} & \mathcal{M}[K+1](X) \\ & \searrow \text{Flrn} & \swarrow \text{Flrn} \\ & & X \end{array} \qquad \begin{array}{ccccc} X^{K+1} & \xrightarrow{\text{acc}} & \mathcal{M}[K+1](X) & \xrightarrow{\text{arr}} & X^{K+1} \\ \text{del} \downarrow & & DD \downarrow & & \downarrow \text{del} \\ X^K & \xrightarrow{\text{acc}} & \mathcal{M}[K](X) & \xrightarrow{\text{arr}} & X^K \end{array}$$

In addition, the rectangles on the right commute.

**Proof.** The above triangle is obtained via the commuting triangle in Lemma 6.1:

$$\text{Flrn} \circ DD \circ \text{acc} = \text{Flrn} \circ \text{acc} \circ \text{del} = \varepsilon \circ \text{del} = \varepsilon = \text{Flrn} \circ \text{acc}.$$

The outer rectangle on the right commutes since it is the rectangle in Lemma 6.1. The inner rectangle on the left commutes by definition (3) of draw-and-delete. Hence the inner rectangle on the right commutes because  $\text{acc}$  is epic.  $\square$

## 7 Sum and zip of multisets

This section introduces two binary operations on multisets, namely the sum and zip. The sum is well-known and involves addition of multiplicities. The zip of multisets is a recently introduced operation (in [24]) that is more complicated. It will be called multizip, to distinguish it from the zip operation for lists. Both operations are obtained basically in the same way, namely by: (1) turning multisets into lists, via arrangement; (2) performing the corresponding operation on lists; (3) turning the result back into a multiset via accumulation.

### 7.1 Summing multisets

Concatenation  $++$  of lists of fixed lengths can be described in a monoidal category as deterministic map of the form:

$$X^K \otimes X^L \xrightarrow[\cong]{++} X^{K+L}.$$

We use it in the following way to define a sum of multisets.

**Definition 7.1** For  $K \in \mathbb{N}$  and  $X \in \mathbf{C}$  define  $+$ :  $\mathcal{M}[K](X) \otimes \mathcal{M}[L](X) \rightarrow \mathcal{M}[K+L](X)$  as composite:

$$+ := \left( \mathcal{M}[K](X) \otimes \mathcal{M}[L](X) \xrightarrow{\text{arr} \otimes \text{arr}} X^K \otimes X^L \xrightarrow[\cong]{++} X^{K+L} \xrightarrow{\text{acc}} \mathcal{M}[K+L](X) \right).$$



Since  $arr$  and  $acc$  are natural, and obviously concatenation  $++$  too, so the composite defining  $+$  in Definition 7.1 is natural too.

**Lemma 7.2** *The sum  $+$  of multisets from Definition 7.1 is commutative and associative, satisfying:*

$$\begin{array}{ccc}
 (\mathcal{M}[K](X) \otimes \mathcal{M}[L](X)) \otimes \mathcal{M}[N](X) & \xrightarrow{+\otimes id} & \mathcal{M}[K+L](X) \otimes \mathcal{M}[N](X) \\
 \downarrow \alpha \cong & & \downarrow + \\
 \mathcal{M}[K](X) \otimes (\mathcal{M}[L](X) \otimes \mathcal{M}[N](X)) & \xrightarrow{id \otimes +} & \mathcal{M}[K](X) \otimes \mathcal{M}[L+N](X) \\
 & & \uparrow + \\
 & & \mathcal{M}[K+L+N](X) \\
 & & \downarrow + \\
 & & \mathcal{M}[K](X) \otimes \mathcal{M}[L+N](X) \\
 & & \downarrow + \\
 & & \mathcal{M}[K](X)
 \end{array}$$

$$\begin{array}{ccc}
 \mathcal{M}[K](X) \otimes \mathcal{M}[L](X) & \xrightarrow{+} & \mathcal{M}[K+L](X) \\
 \downarrow \gamma \cong & & \parallel \\
 \mathcal{M}[L](X) \otimes \mathcal{M}[K](X) & \xrightarrow{+} & \mathcal{M}[L+K](X)
 \end{array}$$

$$\begin{array}{ccc}
 \mathcal{M}[K](X) & \xrightarrow{\cong} & \mathcal{M}[0](X) \otimes \mathcal{M}[K](X) \\
 & \searrow & \downarrow + \\
 & & \mathcal{M}[K](X)
 \end{array}$$

Via this associativity and commutativity of  $+$  we can define an  $K$ -fold sum, for  $n \geq 1$ ,

$$\mathcal{M}[L](X)^K \xrightarrow{\sum_K} \mathcal{M}[K \cdot L](X) \quad \text{and then also} \quad \mathcal{M}[K](\mathcal{M}[L](X)) \xrightarrow{\mu_{K,L}} \mathcal{M}[K \cdot L](X). \quad (4)$$

**Theorem 7.3** *Assume that maps of the form  $acc \otimes acc$  are coequaliser too, e.g. because  $\otimes$  preserves coequalisers.*

- (i) *The sum of multisets  $+$  from Lemma 7.2 satisfies  $+\circ(acc \otimes acc) = acc \circ ++$ , and is thus a (mediating) deterministic map.*
- (ii) *The maps  $\sum_K$  and  $\mu_{K,L}$  in (4) are natural.*
- (iii) *The maps  $\mu_{K,L}$  in (4), together with the maps  $acc[1]: X \rightarrow \mathcal{M}[1](X)$  from Lemma 5.5 (ii), turn  $\mathcal{M}[K]$  into a graded monad, see e.g. [31,20], with respect to the multiplicative monoid  $(\mathbb{N}, \cdot, 1)$  of natural numbers.*

**Proof.** The equation in the first point is easy. It makes  $+$  deterministic, as a mediating map for a deterministic map  $acc \circ ++$ . The second point is obtained by using that the sum  $\sum_K$  and multiplications  $\mu_{K,L}$  maps in (4) are determined by:

$$\begin{array}{ccc}
 (X^L)^K \xrightarrow{acc[L]^K} \mathcal{M}[L](X)^K \xrightarrow{\sum_K} \mathcal{M}[K \cdot L](X) & & \mathcal{M}[L](X)^K \xrightarrow{acc[K]} \mathcal{M}[K](\mathcal{M}[L](X)) \\
 \downarrow ++_K \cong & \nearrow acc[K \cdot L] & \downarrow \mu_{K,L} \\
 X^{K \cdot L} & & \mathcal{M}[K \cdot L](X)
 \end{array} \quad \square$$

When  $(U, m, u)$  is an internal commutative monoid we can define composition maps  $U^K \rightarrow U$  and  $m[K]: \mathcal{M}[K](U) \rightarrow U$ . The latter commutes with the sum in Lemma 7.2:  $m \circ (m[K] \otimes m[L]) = m[K+L] \circ +$ .

## 7.2 Zipping multisets

In functional programming there is the familiar zip operation  $X^K \times Y^K \cong (X \times Y)^K$  that pairs the items of two lists of the same length. It also exists in a monoidal category, via rearrangement:

$$zip := \left( X^K \otimes Y^K = (X \otimes \dots \otimes X) \otimes (Y \otimes \dots \otimes Y) \xrightarrow{\cong} (X \otimes Y) \otimes \dots \otimes (X \otimes Y) = (X \otimes Y)^K \right).$$

Clearly, this  $zip$  is natural in  $X, Y$ . We can now define an analogous zip operation for multisets of the same size, called *multizip*, and written as  $mzip$ . It makes the multiset functor  $\mathcal{M}[K]$  monoidal.

**Definition 7.4** For  $K \in \mathbb{N}$  and  $X, Y \in \mathbf{C}$  define  $mzip: \mathcal{M}[K](X) \otimes \mathcal{M}[K](Y) \rightarrow \mathcal{M}[K](X \times Y)$  as composite:

$$mzip := \left( \mathcal{M}[K](X) \otimes \mathcal{M}[K](Y) \xrightarrow{arr \otimes arr} X^K \otimes Y^K \xrightarrow{zip \cong} (X \otimes Y)^K \xrightarrow{acc} \mathcal{M}[K](X \otimes Y) \right).$$

**Proposition 7.5** (i) *Multizip is natural.*

(ii) *Arrangement commutes with zip and mzip, as in:*

$$\begin{array}{ccc} \mathcal{M}[K](X) \otimes \mathcal{M}[K](Y) & \xrightarrow{\text{arr} \otimes \text{arr}} & X^K \otimes Y^K \\ \text{mzip} \downarrow & & \cong \downarrow \text{zip} \\ \mathcal{M}[K](X \otimes Y) & \xrightarrow{\text{arr}} & (X \otimes Y)^K \end{array}$$

- (iii) *Multizip is associative, making  $\mathcal{M}[K]$  together with the isomorphism  $1 \cong \mathcal{M}[K](1)$  from Lemma 5.5 (ii) a monoidal functor.*
- (iv) *Multizip commutes with projections:  $\mathcal{M}[K](\pi_1) \circ \text{mzip} = \pi_1: \mathcal{M}[K](X) \otimes \mathcal{M}[K](Y) \rightarrow \mathcal{M}[K](X)$ , and similarly for the second projection  $\pi_2$ .*
- (v) *Multizip commutes with draw-and-delete:*

$$\begin{array}{ccc} \mathcal{M}[K+1](X) \otimes \mathcal{M}[K+1](X) & \xrightarrow{DD \otimes DD} & \mathcal{M}[K](X) \otimes \mathcal{M}[K](X) \\ \text{mzip} \downarrow & & \downarrow \text{mzip} \\ \mathcal{M}[K+1](X) & \xrightarrow{DD} & \mathcal{M}[K](X) \end{array}$$

**Proof.** (i) Easy, since all ingredients in the definition of *mzip* are natural.

(ii) Since:

$$\begin{aligned} \text{arr} \circ \text{mzip} &= \text{perm} \circ \text{zip} \circ (\text{arr} \otimes \text{arr}) = \sum_{\sigma \in S_K} \frac{1}{|K|} \cdot (\underline{\sigma} \circ \text{zip} \circ (\text{arr} \otimes \text{arr})) \\ &= \sum_{\sigma \in S_K} \frac{1}{|K|} \cdot (\text{zip} \circ (\underline{\sigma} \otimes \underline{\sigma}) \circ (\text{arr} \otimes \text{arr})) \\ &= \sum_{\sigma \in S_K} \frac{1}{|K|} \cdot (\text{zip} \circ (\text{arr} \otimes \text{arr})) \quad \text{by Lemma 5.4 (iv)} \\ &= \text{zip} \circ (\text{arr} \otimes \text{arr}). \end{aligned}$$

(iii) We reason as follows, using associativity of *zip*, and ignoring monoidal associativity.

$$\begin{aligned} \text{mzip} \circ (\text{mzip} \otimes \text{id}) &= \text{acc} \circ \text{zip} \circ ((\text{arr} \circ \text{mzip}) \otimes \text{arr}) \\ &= \text{acc} \circ \text{zip} \circ (\text{zip} \otimes \text{id}) \circ (\text{arr} \otimes \text{arr} \otimes \text{arr}) \quad \text{by point (ii)} \\ &= \text{acc} \circ \text{zip} \circ (\text{id} \otimes \text{zip}) \circ (\text{arr} \otimes \text{arr} \otimes \text{arr}) \quad \text{by associativity of zip} \\ &= \text{acc} \circ \text{zip} \circ (\text{arr} \otimes (\text{arr} \circ \text{mzip})) \quad \text{by point (ii) again} \\ &= \text{mzip} \circ (\text{id} \otimes \text{mzip}). \end{aligned}$$

(iv) We do the computation for the first projection  $\pi_1: X \otimes Y \rightarrow X$ .

$$\begin{aligned} \mathcal{M}[K](\pi_1) \circ \text{mzip} &= \mathcal{M}[K](\pi_1) \circ \text{acc} \circ \text{zip} \circ (\text{arr} \otimes \text{arr}) = \text{acc} \circ (\pi_1)^K \circ \text{zip} \circ (\text{arr} \otimes \text{arr}) \\ &= \text{acc} \circ \pi_1 \circ (\text{arr} \otimes \text{arr}) \\ &= \text{acc} \circ \text{arr} \circ \pi_1 \\ &= \pi_1 \quad \text{by Lemma 5.4 (iii)}. \end{aligned}$$

(v) Via the following argument:

$$\begin{aligned}
 mzip \circ (DD \otimes DD) &= acc \circ zip \circ ((arr \circ DD) \otimes (arr \circ DD)) \\
 &= acc \circ zip \circ ((del \circ arr) \otimes (del \circ arr)) && \text{by Proposition 6.2} \\
 &= acc \circ zip \circ ((\pi \circ arr) \otimes (\pi \circ arr)) && \text{by Lemma 6.1 (iv)} \\
 &= acc \circ \pi \circ zip \circ (arr \otimes arr) \\
 &= acc \circ \pi \circ arr \circ mzip && \text{by point (ii)} \\
 &= acc \circ del \circ arr \circ acc \circ zip \circ (arr \otimes arr) && \text{by Lemma 6.1 (iv)} \\
 &= DD \circ acc \circ arr \circ acc \circ zip \circ (arr \otimes arr) \\
 &= DD \circ acc \circ zip \circ (arr \otimes arr) \\
 &= DD \circ mzip.
 \end{aligned}$$

□

## 8 Multinomial and hypergeometric distributions

This section finally introduces multinomial and hypergeometric distributions in the current axiomatic setting. The ensuing results are as in [24] for the Kleisli category  $\mathcal{Kl}(\mathcal{D})$  of the distribution monad, but are now obtained in a general categorical setting.

**Definition 8.1** (i) For an arbitrary map  $f: X \rightarrow Y$  and number  $K \in \mathbb{N}$  we define the  $K$ -sized multinomial  $mn[K](f): X \rightarrow \mathcal{M}[K](Y)$  of  $f$  as:

$$mn[K](f) := \left( X \xrightarrow{\delta[K]} X^K \xrightarrow{f^K} Y^K \xrightarrow{acc} \mathcal{M}[K](Y) \right).$$

(ii) For  $L \geq K$  define the hypergeometric map  $hg[L, K]: \mathcal{M}[L](X) \rightarrow \mathcal{M}[K](X)$  via repeated draw-and-delete:

$$hg[L, K] := \underbrace{DD \circ \dots \circ DD}_{L-K \text{ times}} : \mathcal{M}[L](X) \longrightarrow \mathcal{M}[K](X).$$

We first prove several results about multinomials.

**Theorem 8.2** *The multinomial maps satisfy the following properties.*

$$\begin{array}{ccc}
 \begin{array}{ccc} X & \xrightarrow{mn[K](f)} & \mathcal{M}[K](Y) \\ f \downarrow & & \downarrow arr \\ Y & \xrightarrow{\delta[K]} & Y^K \end{array} & & \begin{array}{ccc} X & \xrightarrow{mn[K](f)} & \mathcal{M}[K](Y) \\ & \searrow f & \downarrow Flrn \\ & & Y \end{array} \\
 \\
 \begin{array}{ccc} X & \xrightarrow{mn[K+1](f)} & \mathcal{M}[K+1](Y) \\ & \searrow mn[K](f) & \downarrow DD \\ & & \mathcal{M}[K](Y) \end{array} & & \begin{array}{ccc} X & \xrightarrow{mn[K](mn[L](f))} & \mathcal{M}[K](\mathcal{M}[L](Y)) \\ & \searrow mn[K \cdot L](f) & \downarrow \mu_{K,L} \\ & & \mathcal{M}[K \cdot L](Y) \end{array} \\
 \\
 \begin{array}{ccc} X & \xrightarrow{\delta} X \otimes X & \xrightarrow{mn[K](f) \otimes mn[L](f)} \mathcal{M}[K](Y) \otimes \mathcal{M}[L](Y) \\ & \searrow \mathcal{M}[K+L](f) & \downarrow + \\ & & \mathcal{M}[K+L](Y) \end{array} & & \begin{array}{ccc} X \otimes A & \xrightarrow{mn[K](f) \otimes mn[K](g)} & \mathcal{M}[K](Y) \otimes \mathcal{M}[K](B) \\ & \searrow mn[K](f \otimes g) & \downarrow mzip \\ & & \mathcal{M}[K](Y \otimes B) \end{array}
 \end{array}$$

**Proof.** We handle commutation of the six digrams one by one. The first one follows from Lemma 5.1:

$$arr \circ mn[K](f) = arr \circ acc \circ f^K \circ \delta[K] = perm \circ f^K \circ \delta[K] = f^K \circ perm \circ \delta[K] = f^K \circ \delta[K].$$

Via Lemma 5.2:

$$Flrn \circ mn[K](f) = Flrn \circ acc \circ f^K \circ \delta[K] = \varepsilon[K] \circ f^K \circ \delta[K] = f \circ \varepsilon[K] \circ \delta[K] = f.$$

Next, by Lemma 6.1 (i) and (iii),

$$\begin{aligned} DD \circ mn[K+1](f) &= DD \circ acc \circ f^{K+1} \circ \delta[K+1] \\ &= acc \circ del \circ f^{K+1} \circ \delta[K+1] \\ &= acc \circ f^K \circ del \circ \delta[K+1] = acc \circ f^K \circ \delta[K] = mn[K](f). \end{aligned}$$

Next, we use the diagrams from the proof of Theorem 7.3.

$$\begin{aligned} \mu_{K,L} \circ mn[K](mn[L](f)) &= \mu_{K,L} \circ acc[K] \circ mn[L](f)^K \circ \delta[K] \\ &= \sum_K \circ acc[L]^K \circ (f^L)^K \circ \delta[L]^K \circ \delta[K] \\ &= acc[K \cdot L] \circ ++_K \circ (f^L)^K \circ \delta[L]^K \circ \delta[K] \\ &= acc[K \cdot L] \circ f^{K \cdot L} \circ ++_K \circ \delta[L]^K \circ \delta[K] \\ &= acc[K \cdot L] \circ f^{K \cdot L} \circ \delta[K \cdot L] \\ &= mn[K \cdot L](f). \end{aligned}$$

For the convolution property in the first diagram in the third row:

$$\begin{aligned} + \circ (mn[K](f) \otimes mn[L](f)) \circ \delta &= acc \circ ++ \circ ((arr \circ mn[K](f)) \otimes (arr \circ mn[L](f))) \circ \delta \\ &= acc \circ ++ \circ ((f^K \circ \delta[K]) \otimes (f^L \circ \delta[L])) \circ \delta \\ &= acc \circ f^{K+L} \circ ++ \circ (\delta[K] \otimes \delta[L]) \circ \delta \\ &= acc \circ f^{K+L} \circ \delta[K+L] \\ &= mn[K+L](f). \end{aligned}$$

Finally, along the same lines:

$$\begin{aligned} mzip \circ (mn[K](f) \otimes mn[K](g)) &= acc \circ zip \circ ((arr \circ mn[K](f)) \otimes (arr \circ mn[K](g))) \\ &= acc \circ zip \circ ((f^K \circ \delta[K]) \otimes (g^K \circ \delta[K])) \\ &= acc \circ (f \otimes g)^K \circ zip \circ (\delta[K] \otimes \delta[K]) \\ &= acc \circ (f \otimes g)^K \circ \delta[K] \\ &= mn[K](f \otimes g). \end{aligned} \quad \square$$

We turn to the hypergeometric case. Proofs of the following results are easy, since we have already done the heavy-lifting earlier.

**Theorem 8.3** *The following diagrams about hypergeometric maps commute.*

$$\begin{array}{ccc} \mathcal{M}[L](Y) \xrightarrow{hg[L,K]} \mathcal{M}[K](Y) & & \mathcal{M}[L](X) \xrightarrow{hg[L,K]} \mathcal{M}[K](X) \\ \swarrow mn[L](f) \quad X \quad \searrow mn[K](f) & & \swarrow Flrn \quad X \quad \searrow Flrn \end{array}$$

$$\begin{array}{ccc} \mathcal{M}[L](X) \otimes \mathcal{M}[L](Y) \xrightarrow{hg[L,K] \otimes hg[L,K]} \mathcal{M}[K](X) \otimes \mathcal{M}[K](Y) & & \\ \downarrow mzip & & \downarrow mzip \\ \mathcal{M}[L](X \otimes Y) \xrightarrow{hg[L,K]} \mathcal{M}[K](X \otimes Y) & & \end{array}$$

**Proof.** Commutation of the first triangle, on the left, follows directly from the definition of  $hg[L, K]$ , using the commutation of multinomials with draw-and-delete in Theorem 8.2. Via iterated application of the diagram on the left in Proposition 6.2 one gets commutation of the second triangle, on the right. For the rectangle we use Proposition 7.5 (v).  $\square$

## 9 Concluding remarks

This paper contains some basic handwork in categorical probability, introducing multisets as quotients, with associated multinomial and hypergeometric distributions. It builds on and extends the development of probability theory in Markov categories.

We have not included tensors of multisets, as operation  $\mathcal{M}[K](X) \times \mathcal{M}[L](Y) \rightarrow \mathcal{M}[K \cdot L](X \otimes Y)$ . It is possible to define such an operation, via strength  $st := zip \circ (\delta[L] \otimes id): X \otimes Y^L \rightarrow (X \times Y)^L$  for sequences. When one assumes that coequalisers are preserved by tensors  $\otimes$ , one can define strength for multisets  $mst: X \otimes \mathcal{M}[L](Y) \rightarrow \mathcal{M}[L](X \otimes Y)$  such that  $mst \circ (id \otimes acc) = acc \circ st$ . Although strength for sequences is not commutative, this strength for multisets does satisfy commutativity, in a suitably graded sense. However, the problem is that these strengths, for sequences and for multisets, are not natural, since they involve copying. This generalises the findings in [24] that tensors of multisets are not well-behaved in a probabilistic setting and that the multizip operation should be used instead — for instance because it makes the (fixed-size) multiset functor monoidal and commutes well with multinomial and hypergeometric distributions, as shown here. However, not all is well with multizip, since it does not make  $\mathcal{M}[K]$  into a *monoidal* graded monad. Calculation of a counterexample is quite intimidating.

It remains an interesting question, now with more urgency, what is required to represent other discrete and also continuous distributions in Markov categories.

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## A Appendix

We sketch a proof of Proposition 5.6. Using that  $\otimes$  distributes over  $+$  we formulate the Binomial Theorem as a ‘list-split’ isomorphism *lsplit* in:

$$(X + Y)^K \xrightarrow[\cong]{\text{lsplit}[K]} \bigoplus_{0 \leq i \leq K} \binom{K}{i} \cdot (X^i \otimes Y^{K-i}). \quad (\text{A.1})$$

We use the dot  $\cdot$  for copower, so that  $n \cdot X = X + \dots + X$ . The binomial coefficient  $\binom{K}{i}$  occurs because there are  $\binom{K}{i}$  ways of turning a list of  $X$ ’s of length  $i$  and a list of  $Y$ ’s of length  $K - i$  into a list of  $X + Y$ ’s of length  $K$ , since the alternations of  $X$  and  $Y$  in  $(X + Y)^K$  need to be taken into account.

These *lsplit* isomorphisms in (A.1) are obtained by induction on  $K$ . First, by definition,

$$(X + Y)^0 \cong 1 \cong 1 \otimes 1 \cong 1 \cdot (X^0 \otimes Y^0) \cong \bigoplus_{0 \leq i \leq 0} \binom{0}{i} \cdot (X^i \otimes Y^{0-i}).$$

Next, via the familiar argument, but now in categorical form, using Pascal's identity:

$$\begin{aligned}
(X+Y)^{K+1} &\cong (X+Y) \otimes (X+Y)^K \\
&\cong X \otimes (X+Y)^K + Y \otimes (X+Y)^K \\
&\cong X \otimes \left( \bigoplus_{0 \leq i \leq K} \binom{K}{i} \cdot (X^i \otimes Y^{K-i}) \right) + Y \otimes \left( \bigoplus_{0 \leq i \leq K} \binom{K}{i} \cdot (X^i \otimes Y^{K-i}) \right) \\
&\cong \left( \bigoplus_{0 \leq i \leq K} \binom{K}{i} \cdot (X^{i+1} \otimes Y^{K-i}) \right) + \left( \bigoplus_{0 \leq i \leq K} \binom{K}{i} \cdot (X^i \otimes Y^{K+1-i}) \right) \\
&\cong \binom{K}{0} \cdot (X^1 \otimes Y^K) + \dots + \binom{K}{K} \cdot (X^{K+1} \otimes Y^0) + \binom{K}{0} \cdot (X^0 \otimes Y^{K+1}) + \dots + \binom{K}{K} \cdot (X^K \otimes Y^1) \\
&\cong \binom{K+1}{0} \cdot (X^0 \otimes Y^{K+1}) + \left( \bigoplus_{1 \leq i \leq K} \left( \binom{K}{i-1} + \binom{K}{i} \right) \cdot (X^i \otimes Y^{K+1-i}) \right) + \binom{K+1}{K+1} \cdot (X^{K+1} \otimes Y^0) \\
&\cong \bigoplus_{0 \leq i \leq K+1} \binom{K+1}{i} \cdot (X^i \otimes Y^{K+1-i}).
\end{aligned}$$

A next step is to combine list-split with accumulation.

**Lemma A.1** For  $K \geq 0$  we write  $\text{accs}[K]$  for the sum of cotuples of accumulation maps in:

$$\text{accs}[K] := \left( \bigoplus_{0 \leq i \leq K} \binom{K}{i} \cdot (X^i \otimes Y^{K-i}) \xrightarrow{\bigoplus_{0 \leq i \leq K} [\text{acc}[i] \otimes \text{acc}[K-i]]} \bigoplus_{0 \leq i \leq K} \mathcal{M}[i](X) \otimes \mathcal{M}[K-i](Y) \right)$$

Then:

- (i)  $\text{accs}[K] \circ \text{lsplit} = \bigoplus_{0 \leq i \leq K} \left( (\text{acc}[i] \otimes \text{acc}[K-i]) \circ \nabla \right) \circ \text{lsplit}$ ;
- (ii)  $\text{accs}[K] \circ \text{lsplit} \circ \sigma = \text{accs}[K] \circ \text{lsplit}$  for each permutation  $\sigma \in S_K$ .

**Proof.** The first point says that the the maps  $\text{acc}[i] \otimes \text{acc}[K-i]$  act the same on each of the  $\binom{K}{i}$ -many alternations of  $X$  and  $Y$  in  $(X+Y)^K$ . This follows from an easy combinatorial argument. Similarly for the second point.  $\square$

We are now in a position to define a multiset split map  $\text{msplit}$  in:

$$\begin{array}{ccc}
(X+Y)^K \begin{array}{c} \curvearrowright \\ \vdots \\ \curvearrowleft \end{array} (X+Y)^K & \xrightarrow{\text{acc}} & \mathcal{M}[K](X+Y) \\
& \searrow \text{lsplit} & \downarrow \text{msplit} \\
& \bigoplus_{0 \leq i \leq K} \binom{K}{i} \cdot (X^i \otimes Y^{K-i}) & \downarrow \\
& & \bigoplus_{0 \leq i \leq K} \mathcal{M}[i](X) \otimes \mathcal{M}[K-i](Y)
\end{array} \tag{A.2}$$

Our aim is to show that  $\text{msplit}$  is an isomorphism. There is an obvious map in the reverse direction, which we already write as  $\text{msplit}^{-1}$  in anticipation of the proof. It's define via the sum of multisets from Definition 7.1.

$$\begin{array}{ccc}
\bigoplus_{0 \leq i \leq K} \mathcal{M}[i](X) \otimes \mathcal{M}[K-i](Y) & \xrightarrow{\bigoplus_{0 \leq i \leq K} \mathcal{M}[i](\kappa_1) \otimes \mathcal{M}[K-i](\kappa_2)} & \bigoplus_{0 \leq i \leq K} \mathcal{M}[i](X+Y) \otimes \mathcal{M}[K-i](X+Y) \\
& \searrow \text{msplit}^{-1} & \downarrow [+ ]_{0 \leq i \leq K} \\
& & \mathcal{M}[K](X+Y)
\end{array} \tag{A.3}$$

It is now "obvious" that  $\text{msplit}$  and  $\text{msplit}^{-1}$  are each other's inverses, proving Proposition 5.6 (i).

We add a proof of Proposition 5.6 (ii), stating that  $\mathcal{M}[K](\underline{n}) \cong \binom{n}{K} \cdot 1$ , where the multichoose coefficient is defined as  $\binom{n}{K} = \binom{n+K-1}{K}$ . This result is obtained by induction on  $n \geq 1$ . For  $n = 1$  we get, by Lemma 5.5 (iii):

$$\mathcal{M}[K](\underline{1}) = \mathcal{M}[K](1) \cong 1 = \binom{1+K-1}{K} \cdot 1 = \binom{1}{K} \cdot 1.$$

Next,

$$\begin{aligned} \mathcal{M}[K](\underline{n+1}) \cong \mathcal{M}[K](\underline{n} + 1) &\cong \bigoplus_{0 \leq i \leq K} \mathcal{M}[i](\underline{n}) \otimes \mathcal{M}[K-i](1) \text{ by Proposition 5.6 (i)} \\ &\cong \bigoplus_{0 \leq i \leq K} \left[ \binom{n}{i} \cdot 1 \right] \otimes 1 \quad \text{by induction hypothesis, and Lemma 5.5 (iii)} \\ &\cong \left[ \sum_{0 \leq i \leq K} \binom{n}{i} \right] \cdot 1 \\ &= \binom{n+1}{K} \cdot 1. \end{aligned}$$

The latter equation is a basic property of multichoose.