# Sharp Elements and the Scott Topology of Continuous Dcpos

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**Abstract**

Working constructively, we study continuous directed complete posets (dcpos) and the Scott topology. Our two primary novelties are a notion of intrinsic apartness and a notion of sharp elements. Being apart is a positive formulation of being unequal, similar to how inhabitedness is a positive formulation of nonemptiness. To exemplify sharpness, we note that a lower real is sharp if and only if it is located. Our first main result is that for a large class of continuous dcpos, the Bridges–Vîţă apartness topology and the Scott topology coincide. Although we cannot expect a tight or cotransitive apartness on nontrivial dcpos, we prove that the intrinsic apartness is both tight and cotransitive when restricted to the sharp elements of a continuous dcpo. These include the strongly maximal elements, as studied by Smyth and Heckmann. We develop the theory of strongly maximal elements highlighting its connection to sharpness and the Lawson topology. Finally, we illustrate the intrinsic apartness, sharpness and strong maximality by considering several natural examples of continuous dcpos: the Cantor and Baire domains, the partial Dedekind reals and the lower reals.

*Keywords:* constructivity, apartness, sharpness, Scott topology, continuous dcpo, basis, domain, strong maximality, Lawson topology

## 1 Introduction

Domain theory [2] is rich with applications in semantics of programming languages [32,31,27], topology and algebra [13], and higher-type computability [21]. The basic objects of domain theory are directed complete posets (dcpos), although we often restrict our attention to algebraic or continuous dcpos which are generated by so-called compact elements or, more generally, by the so-called way-below relation (Section 2). We examine the Scott topology on dcpos using an apartness relation and a notion of sharp elements. Our work is constructive in the sense that we do not assume the principle of excluded middle or choice axioms, so our results are valid in any elementary topos.

Classically, i.e. when assuming excluded middle, a dcpo with the Scott topology satisfies $T_0$-separation: if two points have the same Scott open neighbourhoods, then they are equal. This holds constructively if we restrict to continuous dcpos. A classically equivalent formulation of $T_0$-separation is: if $x \neq y$, then there is a Scott open separating $x$ and $y$, i.e. containing $x$ but not $y$ or vice versa. This second formulation is equivalent to excluded middle. This brings us to the first main notion of this paper (Section 3). We say that $x$ and $y$ are intrinsically apart, written $x \mathrel{\#} y$, if there is a Scott open containing $x$ but not $y$ or vice versa. Then $x \mathrel{\#} y$ is a positive formulation of $x \neq y$, similar to how inhabitedness (i.e. $\exists\, x \in X$) is a positive formulation of nonemptiness (i.e. $X \neq \emptyset$).

This definition works for any dcpo, but the intrinsic apartness is mostly of interest to us for continuous dcpos. In fact, the apartness really starts to gain traction for continuous dcpos that have a basis satisfying certain decidability conditions. For example, we prove that for such continuous dcpos, the Bridges–Vîţă apartness topology [7] and the Scott topology coincide (Section 4). Thus our work may be regarded as showing

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that the constructive framework by Bridges and Vîţă is applicable to domain theory. It should be noted that these decidability conditions are satisfied by the major examples in applications of domain theory to topology and computation. Moreover, these conditions are stable under products of dcpos and, in the case of bounded complete algebraic dcpos, under exponentials (Section 2).

In [6, p. 7], [28, p. 8] and [7, p. 8], an irreflexive and symmetric relation is called an inequality (relation) and the symbol $\neq$ is used to denote it. In [5, Definition 2.1], an inequality is moreover required to be cotransitive:

$$\text{if } x \neq y, \text{ then } x \neq z \text{ or } y \neq z \text{ for any } x, y \text{ and } z.$$

The latter is called a preapartness in [36, Section 8.1.2] and the symbol $\#$ is used to denote it, reserving $\neq$ for the logical negation of equality and the word apartness for a relation that is also tight: if $\neg(x \mathrel{\#} y)$, then $x = y$.

**Warning 1.1** We deviate from the above and use the word apartness and the symbol $\#$ for an irreflexive and symmetric relation, so we do not require it to be cotransitive or tight.

The reasons for our choice of terminology and notations are as follows: (i) we wish to reserve $\neq$ for the negation of equality as in [36, Section 8.1.2]; (ii) the word inequality is confusingly also used in the context of posets to refer to the partial order; and finally, (iii) the word inequality seems to suggest that the negation of the inequality relation is an equivalence relation, but, in the absence of cotransitivity, it need not be.

Actually, we prove that no apartness on a nontrivial dcpo can be cotransitive or tight unless (weak) excluded middle holds. However, there is a natural collection of elements for which the intrinsic apartness is both tight and apartness: the sharp elements (Section 5). Sharpness is slightly involved in general, but it is easy to understand for algebraic dcpos: an element $x$ is sharp if and only if for every compact element it is decidable whether it is below $x$. Moreover, the notion is quite natural in many examples. For instance, the sharp elements of a powerset are exactly the decidable subsets and the sharp lower reals are precisely the located ones.

An import class of sharp elements is given by the strongly maximal elements (Section 6). These were studied in a classical context by Smyth [34] and Heckmann [14], because of their desirable properties. For instance, while the subspace of maximal elements may fail to be Hausdorff, the subspace of strongly maximal elements is both Hausdorff (two distinct points can be separated by disjoint Scott opens) and regular (every neighbourhood contains a Scott closed neighbourhood). As shown in [34], strong maximality is closely related to the Lawson topology. Specifically, Smyth proved that a point $x$ is strongly maximal if and only if every Lawson neighbourhood of $x$ contains a Scott neighbourhood of $x$. Using sharpness, we offer a constructive proof of this.

Finally, we illustrate (Section 7) the above notions by presenting examples of continuous dcpos that embed well-known spaces as strongly maximal elements: the Cantor and Baire domains, the partial Dedekind reals and the lower reals.

### Related work

There are numerous accounts of basic domain theory in several constructive systems, such as [30,24,25,22,17,18] in the predicative setting of formal topology [29,8], as well as works in various type theories: [15] in (a version of) Martin-Löf Type Theory, [20] in Agda, [4,11] in Coq and our previous work [9] in univalent foundations. Besides that, the papers [3,26] are specifically aimed at program extraction.

Our work is not situated in formal topology and we work informally in (impredicative) set theory without using excluded middle or choice axioms. We also consider completeness with respect to all directed subsets and not just $\omega$-chains as is done in [3,26]. The principal contributions of our work are the aforementioned notions of intrinsic apartness and sharp elements, although the idea of sharpness also appears in formal topology: an element of a continuous dcpo is sharp if and only if its filter of Scott open neighbourhoods is located in the sense of Spitters [35] and Kawai [17].

If, as advocated in [1,37,33], we think of (Scott) opens as observable properties, then this suggests that we label two points as apart if we have made conflicting observations about them, i.e. if there are disjoint opens separating the points. Indeed, (an equivalent formulation of) this notion is used in Smyth's [34, p. 362]. While these notions are certainly useful, both in the presence and absence of excluded middle, our apartness serves a different purpose: It is a positive formulation of the negation of equality used when reasoning about the Scott topology on a dcpo, which (classically) is only a $T_0$-space that isn't Hausdorff in general. By contrast, an apartness based on disjoint opens would supposedly perform a similar job for a Hausdorff space, such as a dcpo with the Lawson topology.

Finally, von Plato [38] gives a constructive account of so-called positive partial orders: sets with a binary relation $\not\leq$ that is irreflexive and cotransitive (i.e. if $x \not\leq y$, then $x \not\leq z$ or $y \not\leq z$ for any elements $x$, $y$ and $z$). Our notion $\not\sqsubseteq$ from Definition 3.1 bears some similarity, but our work is fundamentally different for two reasons. Firstly, $\not\sqsubseteq$ is not cotransitive. Indeed, we cannot expect such a cotransitive relation on nontrivial dcpos, cf. Theorem 5.6. Secondly, in [38] equality is a derived notion from $\not\leq$, while equality is primitive for us.

## 2 Preliminaries

We give the basic definitions and results in the theory of (continuous) dcpos. It is not adequate to simply refer the reader to classical texts on domain theory [2,13], because two classically equivalent definitions need not be constructively equivalent, and hence we need to make choices here. For example, while classically every Scott open subset is the complement of a Scott closed subset, this does not hold constructively (Lemma 2.9). The results presented here are all provable constructively. Constructive proofs of standard results, such as Lemma 2.3 and Proposition 2.17 (the interpolation property), can be found in [9]. Finally, in Section 2.3 we introduce and study some decidability conditions on bases of dcpos that will make several appearances throughout the paper. These decidability conditions always hold if excluded middle is assumed.

**Definition 2.1**

(i) A subset $S$ of a poset $(X, \sqsubseteq)$ is *directed* if it is *inhabited*, meaning there exists $s \in S$, and *semidirected*: for every two points $x, y \in S$ there exists $z \in S$ with $x \sqsubseteq z$ and $y \sqsubseteq z$.

(ii) A *directed complete poset (dcpo)* is a poset where every directed subset $S$ has a supremum, denoted by $\bigsqcup S$.

(iii) A dcpo is *pointed* if it has a least element, typically denoted by $\bot$.

Notice that a poset is a pointed dcpo if and only if it has suprema for all semidirected subsets. In fact, given a pointed dcpo $D$ and a semidirected subset $S \subseteq D$, we can consider the directed subset $S \cup \{\bot\}$ of $D$ whose supremum is also the supremum of $S$.

**Definition 2.2** An element $x$ of a dcpo $D$ is *way below* an element $y \in D$ if for every directed subset $S$ with $y \sqsubseteq \bigsqcup S$ there exists $s \in S$ such that $x \sqsubseteq s$ already. We denote this by $x \ll y$ and say $x$ is *way below* $y$.

The following is easily verified.

**Lemma 2.3** *The way-below relation enjoys the following properties:*

(i) *it is transitive;*

(ii) *if $x \sqsubseteq y \ll z$, then $x \ll z$ for every $x$, $y$ and $z$;*

(iii) *if $x \ll y \sqsubseteq z$, then $x \ll z$ for every $x$, $y$ and $z$.*

**Definition 2.4** A dcpo $D$ is *continuous* if for every element $x \in D$, the subset $\twoheaddownarrow x :\equiv \{y \in D \mid y \ll x\}$ is directed and its supremum is $x$.

**Definition 2.5** An element of a dcpo is *compact* if it is way below itself. A dcpo $D$ is *algebraic* if for every element $x \in D$, the subset $\{c \in D \mid c \sqsubseteq x \text{ and } c \text{ is compact}\}$ is directed with supremum $x$.

**Proposition 2.6** *Every algebraic dcpo is continuous.*

### *2.1 The Scott Topology*

**Definition 2.7**

(i) A subset $C$ of a dcpo $D$ is *Scott closed* if it is closed under directed suprema and a lower set: if $x \sqsubseteq y \in C$, then $x \in C$ too.

(ii) A subset $U$ of a dcpo $D$ is *Scott open* if it is an upper set and for every directed subset $S \subseteq D$ with $\bigsqcup S \in U$, there exists $s \in S$ such that $s \in U$ already.

**Example 2.8** For any element $x$ of a dcpo $D$, the subset $\downarrow x :\equiv \{y \in D \mid y \sqsubseteq x\}$ is Scott closed. If $D$ is continuous, then the subset $\twoheaduparrow x :\equiv \{y \in D \mid x \ll y\}$ is Scott open. (One proves this using the interpolation property, which is Proposition 2.17 below.) Moreover, if $D$ is continuous, then the set $\{\twoheaduparrow x \mid x \in X\}$ is a basis for the Scott topology on $D$.

**Lemma 2.9** *The complement of a Scott open subset is Scott closed. The converse holds if and only if excluded middle does, as we prove in Proposition 3.9.*

**Definition 2.10** In a topological space $X$, the *interior* of a subset $S \subseteq X$ is the largest open of $X$ contained in $S$. Dually, the *closure* of a subset $S \subseteq X$ is the smallest closed subset of $X$ that contains $S$.

### *2.2 (Abstract) Bases*

**Definition 2.11** A *basis* for a dcpo $D$ is a subset $B \subseteq D$ such that for every element $x \in D$, the subset $B \cap \twoheaddownarrow x$ is directed with supremum $x$.

**Lemma 2.12** *A dcpo is continuous if and only if it has a basis and a dcpo is algebraic if and only if it has a basis of compact elements. Moreover, if $B$ is a basis for an algebraic dcpo $D$, then $B$ must contain every compact element of $D$. Hence, an algebraic dcpo has a unique smallest basis consisting of compact elements.*

**Example 2.13** The powerset $\mathcal{P}(X)$ of any set $X$ ordered by inclusion and with suprema given by unions is a pointed algebraic dcpo. Its compact elements are the Kuratowski finite subsets of $X$. A set $X$ is *Kuratowski finite* if it is finitely enumerable, i.e. there exists a surjection $e \colon \{0, \ldots, n-1\} \twoheadrightarrow X$ for some number $n \in \mathbb{N}$.

**Definition 2.14** The *Sierpiński domain* $\mathbb{S}$ is the free pointed dcpo on a single generator. We can realize $\mathbb{S}$ as the set of truth values, i.e. as the powerset $\mathcal{P}(\{*\})$ of a singleton. The compact elements of $\mathbb{S}$ are exactly the elements $\bot :\equiv \emptyset$ and $\top :\equiv \{*\}$.

**Lemma 2.15** *For every dcpo $D$, a subset $B \subseteq D$ is a basis for the dcpo $D$ if and only if $\{\twoheaduparrow b \mid b \in B\}$ is a basis for the Scott topology on $D$.*

**Proposition 2.16** *Every basis of a continuous dcpo is dense with respect to the Scott topology in the following (classically equivalent) ways:*

(i) *the Scott closure of the basis is the whole dcpo;*

(ii) *every inhabited Scott open contains a point in the basis.*

**Proposition 2.17 (Interpolation)** *If $x \ll y$ are elements of a continuous dcpo $D$, then there exists $b \in D$ with $x \ll b \ll y$. Moreover, if $D$ has a basis $B$, then there exists such an element $b$ in $B$.*

**Lemma 2.18** *For every two elements $x$ and $y$ of a continuous dcpo $D$ we have $x \sqsubseteq y$ if and only if $\forall_{z \in D}\,(z \ll x \to z \ll y)$. Moreover, if $D$ has a basis $B$, then $x \sqsubseteq y$ if and only if $\forall_{b \in B}\,(b \ll x \to b \ll y)$.*

**Definition 2.19** An *abstract basis* is a pair $(B, \prec)$ such that $\prec$ is transitive and *interpolative*: for every $b \in B$, the subset $\twoheaddownarrow b :\equiv \{a \in B \mid a \prec b\}$ is directed. The *rounded ideal completion* $\mathrm{Idl}(B, \prec)$ of an abstract basis $(B, \prec)$ consists of directed lower sets of $(B, \prec)$, known as *(rounded) ideals*, ordered by subset inclusion. It is a continuous dcpo with basis $\{\twoheaddownarrow b \mid b \in B\}$ and directed suprema given by unions.

**Lemma 2.20** *If the relation $\prec$ of an abstract basis $(B, \prec)$ is reflexive, then $\mathrm{Idl}(B, \prec)$ is algebraic and its compact elements are exactly those of the form $\twoheaddownarrow b$ for $b \in B$.*

### *2.3 Decidability Conditions*

Every continuous dcpo $D$ has a basis, namely $D$ itself. Our interest in bases lies in the fact that we can ask a dcpo to have a basis satisfying certain decidability conditions that we couldn't reasonably impose on the entire dcpo. For instance, the basis $\{\bot, \top\}$ of the Sierpiński domain $\mathbb{S}$ has decidable equality, but decidable equality on all of $\mathbb{S}$ is equivalent to excluded middle.

The first decidability condition that we will consider is for bases $B$ of a pointed continuous dcpo:

$$\text{For every } b \in B, \text{ it is decidable whether } b = \bot. \tag{$\delta_\bot$}$$

The second and third decidability conditions are for bases of any continuous dcpo:

$$\text{For every } a, b \in B, \text{ it is decidable whether } a \ll b. \tag{$\delta_\ll$}$$

$$\text{For every } a, b \in B, \text{ it is decidable whether } a \sqsubseteq b. \tag{$\delta_\sqsubseteq$}$$

Observe that each of ($\delta_\sqsubseteq$) and ($\delta_\ll$) implies ($\delta_\bot$) for pointed dcpos, because $\bot$ is compact. In general, neither of the conditions ($\delta_\sqsubseteq$) and ($\delta_\ll$) implies the other. However, in some cases, for example when $B$ is finite, ($\delta_\ll$) is a weaker condition, because of Lemma 2.18. Moreover, if the dcpo is algebraic then conditions ($\delta_\sqsubseteq$) and ($\delta_\ll$) are equivalent for the unique basis of compact elements.

Finally, we remark that many natural examples in domain theory satisfy the decidability conditions. In particular, this holds for all examples in Section 7.

**Proposition 2.21** *Let $D$ be any pointed dcpo that is nontrivial in the sense that there exists $x \in D$ with $x \neq \bot$. If $y = \bot$ is decidable for every $y \in D$, then weak excluded middle follows.*

Moreover, if the order relation of a dcpo is decidable, then, by antisymmetry, the dcpo must have decidable equality, but we showed in [10, Corollary 39] that this implies (weak) excluded middle, unless the dcpo is trivial.

### 2.3.1 Closure Under Products and Exponentials

**Definition 2.22** The *product* $D \times E$ of two dcpos $D$ and $E$ is given by their Cartesian product ordered pairwise. The supremum of a directed subset $S \subseteq D \times E$ is given by the pair of suprema $\bigsqcup\{x \in D \mid \exists_{y \in E}\,(x,y) \in S\}$ and $\bigsqcup\{y \in D \mid \exists_{x \in D}\,(x,y) \in S\}$.

**Proposition 2.23** *If $D$ and $E$ are continuous dcpos with bases $B_D$ and $B_E$, then $B_D \times B_E$ is a basis for the product $D \times E$. Also, if $B_D$ and $B_E$ both satisfy ($\delta_\perp$), then so does $B_D \times B_E$, and similarly for ($\delta_\ll$) and ($\delta_\sqsubseteq$).*

**Definition 2.24** A function between dcpos is *Scott continuous* if it preserves directed suprema.

**Definition 2.25** The *exponential* $E^D$ of two dcpos $D$ and $E$ is given by the set of Scott continuous functions from $D$ to $E$ ordered pointwise, i.e. $f \sqsubseteq g$ if $\forall_{x \in D}\, f(x) \sqsubseteq g(x)$ for $f, g \colon D \to E$. Suprema of directed subsets are also given pointwise.

**Definition 2.26** Given an element $x$ of a dcpo $D$ and an element $y$ of a pointed dcpo $E$, the *single-step function* $(\!|x \Rightarrow y|\!) \colon D \to E$ is defined as $(\!|x \Rightarrow y|\!)(d) :\equiv \bigsqcup\{y \mid x \sqsubseteq d\}$. A *step-function* is the supremum of a Kuratowski finite (recall Example 2.13) subset of single-step functions.

**Lemma 2.27** *If $x$ is a compact element of $D$ and $y$ is any element of a pointed dcpo $E$, then $(\!|x \Rightarrow y|\!) \colon D \to E$ is Scott continuous. If $y$ is also compact, then $(\!|x \Rightarrow y|\!)$ is a compact element of the exponential $E^D$.*

**Lemma 2.28** *For every element $x$ of a dcpo $D$, element $y$ of a pointed dcpo $E$ and Scott continuous function $f \colon D \to E$, we have $(\!|x \Rightarrow y|\!) \sqsubseteq f$ if and only if $y \sqsubseteq f(x)$.*

**Lemma 2.29** *The compact elements of a dcpo are closed under existing finite suprema.*

**Definition 2.30** A subset $S$ of a poset $(X, \sqsubseteq)$ is *bounded* if there exists $x \in X$ such that $s \sqsubseteq x$ for every $s \in S$. A poset $(X, \sqsubseteq)$ is *bounded complete* if every bounded subset $S$ of $X$ has a supremum $\bigsqcup S$ in $X$.

**Proposition 2.31** *If $D$ is an inhabited algebraic dcpo with basis of compact elements $B_D$ and $E$ is a pointed bounded complete algebraic dcpo with basis of compact elements $B_E$, then*

$$B :\equiv \Big\{ \bigsqcup S \mid S \textit{ is a bounded Kuratowski finite subset of single-step functions} \\ \textit{of the form } (\!|a \Rightarrow b|\!) \textit{ with } a \in B_D \textit{ and } b \in B_E \Big\}.$$

*is the basis of compact elements for the algebraic exponential $E^D$. Moreover, if $B_E$ satisfies ($\delta_\perp$), then so does $B$. Finally, if $B_D$ and $B_E$ both satisfy ($\delta_\sqsubseteq$) (or equivalently, ($\delta_\ll$)), then $B$ satisfies ($\delta_\sqsubseteq$) and ($\delta_\ll$) too.*

## 3 The Intrinsic Apartness

**Definition 3.1** The *specialization preorder* on a topological space $X$ is the preorder $\leq$ on $X$ given by putting $x \leq y$ if every open neighbourhood of $x$ is an open neighbourhood of $y$. Given $x, y \in X$, we write $x \not\leq y$ and say that $y$ *does not specialize* $x$ if there exists an open neighbourhood of $x$ that does not contain $y$.

Observe that $x \not\leq y$ is classically equivalent to $x \nleq y$, the logical negation of $x \leq y$. We also write $x \not\sqsubseteq y$ for the logical negation of $x \sqsubseteq y$.

**Definition 3.2** An *apartness* on a set $X$ is a binary relation $\#$ on $X$ satisfying

(i) *irreflexivity*: $x \# x$ is false for every $x \in X$;

(ii) *symmetry*: if $x \# y$, then $y \# x$ for every $x, y \in X$.

If $x \# y$ holds, then $x$ and $y$ are said to be *apart*.

Notice that we do not require cotransitivity or tightness, cf. Warning 1.1. Notice that irreflexivity implies that if $x \# y$, then $x \neq y$, so $\#$ is a strengthening of inequality.

**Definition 3.3** Given two points $x$ and $y$ of a topological space $X$, we say that $x$ and $y$ are *intrinsically apart*, written $x \# y$, if $x \not\leq y$ or $y \not\leq x$. Thus, $x$ is intrinsically apart from $y$ if there is a Scott open neighbourhood of $x$ that does not contain $y$ or vice versa. It is clear that the relation $\#$ is an apartness in the sense of Definition 3.2.

With excluded middle, one can show that the specialization preorder for the Scott topology on a dcpo coincides with the partial order of the dcpo. In particular, the specialization preorder is in fact a partial order. Constructively, we still have the following result.

**Lemma 3.4** *Let $x$ and $y$ be elements of a dcpo $D$. If $x \sqsubseteq y$, then $x \leq y$, where $\leq$ is the specialization order of the Scott topology. If $D$ is continuous, then the converse holds too, so $\sqsubseteq$ and $\leq$ coincide in that case.*

**Lemma 3.5** *For a continuous dcpo $D$ we have $x \not\sqsubseteq y$ if and only if there exists $b \in D$ such that $b \ll x$, but $b \not\sqsubseteq y$. Moreover, if $D$ has a basis $B$, then there exists such an element $b$ in $B$.*

The condition in Lemma 3.5 appears in a remark right after [13, Definition I-1.6], as a classically equivalent reading of $x \not\sqsubseteq y$.

**Example 3.6** Consider the powerset $\mathcal{P}(X)$ of a set $X$ as a pointed algebraic dcpo. Using Lemma 3.5, we see that a subset $A \in \mathcal{P}(X)$ is intrinsically apart from the empty set if and only if $A$ is inhabited. More generally, we have $A \not\sqsubseteq B$ if and only if $B \setminus A$ is inhabited for every two subsets $A, B \in \mathcal{P}(X)$.

**Proposition 3.7**

(i) *For any elements $x$ and $y$ of a dcpo $D$, we have that $x \not\sqsubseteq y$ implies $x \not\sqsubseteq y$.*

(ii) *The converse of* (i) *holds if and only if excluded middle holds. In particular, if the converse of* (i) *holds for all elements of the Sierpiński domain $\mathbb{S}$, then excluded middle follows.*

(iii) *For any elements $x$ and $y$ of a dcpo $D$, we have that $x \# y$ implies $x \neq y$.*

(iv) *The converse of* (iii) *holds if and only if excluded middle holds. In particular, if the converse of* (iii) *holds for all elements of the Sierpiński domain $\mathbb{S}$, then excluded middle follows.*

(v) *If $c$ is a compact element of a dcpo $D$ and $x \in D$, then $c \not\sqsubseteq x$ implies $c \not\sqsubseteq x$, without the need to assume excluded middle.*

With excluded middle, complements of Scott closed subsets are Scott open. In particular, $\{x \in D \mid x \not\sqsubseteq y\}$ is Scott open for any element $y$ of a dcpo $D$. Constructively, we have the following result.

**Proposition 3.8** *For any element $y$ of a dcpo $D$, the Scott interior of $\{x \in D \mid x \not\sqsubseteq y\}$ is given by the subset $\{x \in D \mid x \not\sqsubseteq y\}$, where we recall that $x \not\sqsubseteq y$ means that there exists a Scott open containing $x$ but not $y$.*

**Proposition 3.9** *If the complement of every Scott closed subset of the Sierpiński domain $\mathbb{S}$ is Scott open, then excluded middle follows.*

An element $x$ of a pointed dcpo may be said to be nontrivial if $x \neq \bot$. Given our notion of apartness, we might consider the constructively stronger $x \# \bot$. We show that this is related to Johnstone's notion of positivity [16, p. 98]. In [10, Definition 25] we adapted Johnstone's positivity from locales to a general class of posets that includes dcpos. Here we give an equivalent, but simpler, definition just for pointed dcpos.

**Definition 3.10** An element $x$ of a pointed dcpo $D$ is *positive* if every semidirected subset $S \subseteq D$ satisfying $x \sqsubseteq \bigsqcup S$ is inhabited (and hence directed).

**Proposition 3.11** *For every element $x$ of a pointed dcpo, if $x \# \bot$, then $x$ is positive. In the other direction, if $D$ is a continuous pointed dcpo with a basis satisfying ($\delta_\bot$), then every positive element of $D$ is apart from $\bot$.*

## 4 The Apartness Topology

In [7, Section 2.2], Bridges and Vîţă start with a topological space $X$ equipped with an apartness relation $\#$ and, using the topology and apartness, define a second topology on $X$, known as the *apartness topology*. A natural question is whether the original topology and the apartness topology coincide. For example, if $X$ is a metric space and we set two points of $X$ to be apart if their distance is strictly positive, then the Bridges-Vîţă apartness topology and the topology induced by the metric coincide [7, Proposition 2.2.10]. We show, assuming a modest $\neg\neg$-stability condition that holds in examples of interest, that the Scott topology on a continuous dcpo with the intrinsic apartness relation coincides with the apartness topology. We start by repeating some basic definitions and results of Bridges and Vîţă. Recalling Warning 1.1, we remind the reader familiar with [7] that Bridges and Vîţă write $\neq$ and use the word *inequality* for what we denote by $\#$ and call *apartness*.

In constructive mathematics, positively defined notions are usually more useful than negatively defined ones. We already saw examples of this: $\#$ versus $\neq$ and $\not\sqsubseteq$ versus $\not\sqsubseteq$. We now use an apartness to give a positive definition of the complement of a set.

**Definition 4.1** Given a subset $A$ of a set $X$ with an apartness $\#$ we define the *logical complement* and the *complement* respectively as

(i) $\neg A := \{x \in X \mid x \notin A\} = \{x \in X \mid \forall_{y \in A}\, x \neq y\}$;

(ii) $\sim A := \{x \in X \mid \forall_{y \in A}\, x \# y\}$.

**Definition 4.2** A topological space $X$ equipped with an apartness $\#$ satisfies the *(topological) reverse Kolmogorov property* if for every open $U$ and points $x, y \in X$ with $x \in U$ and $y \notin U$, we have $x \# y$.

**Lemma 4.3 (Proposition 2.2.2 in [7])** *If a topological space $X$ equipped with an apartness $\#$ satisfies the reverse Kolmogorov property, then for every subset $A \subseteq X$, we have $(\neg A)^\circ = (\sim A)^\circ$, where $Y^\circ$ denotes the interior of $Y$ in $X$.*

**Definition 4.4** For an element $x$ of a topological space $X$ with an apartness $\#$ and a subset $A \subseteq X$, we write $x \bowtie A$ if $x \in (\sim A)^\circ$. This gives rise to the *apartness complement*: $-A := \{x \in X \mid x \bowtie A\}$. Subsets of the form $-A$ are called *nearly open*. The *apartness topology* on $X$ is the topology whose basic opens are the nearly open subsets of $X$.

**Lemma 4.5 (Proposition 2.2.7 in [7])** *Every nearly open subset of $X$ is open in the original topology of $X$.*

The following are original contributions.

**Definition 4.6** Say that a basis $B$ for a topological space $X$ is *$\neg\neg$-stable* if $U = (\neg\neg U)^\circ$ for every open $U \in B$. Note that $U \subseteq (\neg\neg U)^\circ$ holds for every open $U$, so the relevant condition is that $(\neg\neg U)^\circ \subseteq U$ for every basic open $U$.

Examples of such bases will be provided by Theorem 4.8 below.

**Lemma 4.7** *If a topological space $X$ equipped with an apartness $\#$ satisfies the reverse Kolmogorov property and has a $\neg\neg$-stable basis, then the original topology on $X$ and the apartness topology on $X$ coincide, i.e. a subset of $X$ is open (in the original topology) if and only if it is nearly open.*

**Theorem 4.8** *Let $D$ be a continuous dcpo with a basis $B$. Each of the following conditions on the basis $B$ implies that $\{\Uparrow b \mid b \in B\}$ is a $\neg\neg$-stable basis for the Scott topology on $D$:*

(i) *For every $a, b \in B$, if $\neg\neg(a \ll b)$, then $(a \ll b)$.*

(ii) *For every $a, b \in B$, if $\neg\neg(a \sqsubseteq b)$, then $(a \sqsubseteq b)$.*

(iii) *$B$ satisfies $(\delta_{\ll})$*

(iv) *$B$ satisfies $(\delta_{\sqsubseteq})$*

*Hence, if one of these conditions holds, then the Scott topology on $D$ coincides with the apartness topology on $D$ with respect to the intrinsic apartness, i.e. a subset of $D$ is Scott open if and only if it is nearly open.*

**Proof.** The final claim follows from Lemma 4.7 and the fact that the intrinsic apartness satisfies the reverse Kolmogorov property (by definition). Moreover, (iii) implies (i) and (iv) implies (ii). So it suffices to show that if (i) or (ii) holds, then $\{\Uparrow a \mid a \in B\}$ is a $\neg\neg$-stable basis for the Scott topology on $D$, viz. that $(\neg\neg \Uparrow a)^\circ \subseteq \Uparrow a$ for every $a \in B$. Let $a \in B$ be arbitrary and suppose that $x \in (\neg\neg \Uparrow a)^\circ$. Using Scott openness and continuity of $D$, there exists $b \in B$ such that $b \ll x$ and $b \in \neg\neg \Uparrow a$. The latter just says that $\neg\neg(a \ll b)$. So if condition (i) holds, then we get $a \ll b$, so $a \ll x$ and $x \in \Uparrow a$, as desired. Now suppose that condition (ii) holds. From $\neg\neg(a \ll b)$, we get $\neg\neg(a \sqsubseteq b)$ and hence, $a \sqsubseteq b$ by condition (ii). So $a \sqsubseteq b \ll x$ and $x \in \Uparrow a$, as wished. □

## 5 Tightness, Cotransitivity and Sharpness

**Definition 5.1** An apartness relation $\sharp$ on a set $X$ is *tight* if $\neg(x \sharp y)$ implies $x = y$ for every $x, y \in X$. The apartness is *cotransitive* if $x \sharp y$ implies the disjunction of $x \sharp z$ and $x \sharp y$ for every $x, y, z \in X$.

**Lemma 5.2** *If $X$ is a set with a tight apartness, then $X$ is $\neg\neg$-separated, viz. $\neg\neg(x = y)$ implies $x = y$ for every $x, y \in X$.*

It will be helpful to employ the following positive (but classically equivalent) formulation of $(x \sqsubseteq y) \wedge (x \neq y)$ from [10, Definition 20].

**Definition 5.3** An element $x$ of a dcpo $D$ is *strictly below* an element $y$, written $x \sqsubset y$, if $x \sqsubseteq y$ and for every $z \sqsupseteq y$ and proposition $P$, the equality $z = \bigsqcup(\{x\} \cup \{z \mid P\})$ implies $P$.

We can relate the above notion to the intrinsic apartness, but, although we do not have a counterexample, we believe $x \sqsubset y$ to be weaker than $x \# y$ in general.

**Proposition 5.4** *If $x \sqsubseteq y$ are elements of a dcpo $D$ with $x \# y$, then $x \sqsubset y$.*

**Example 5.5** In the Sierpiński domain $\mathbb{S}$ we have $\bot \sqsubset \top$. In the powerset $\mathcal{P}(X)$ of some set $X$, the empty set is strictly below a subset $A$ of $X$ if and only if $A$ is inhabited. More generally, if $A \subseteq B$ are subsets of some set, then $A \sqsubset B$ holds if $B \setminus A$ is inhabited, and if $A$ is a decidable subset and $A \sqsubset B$, then $B \setminus A$ is inhabited.

The following shows that we cannot expect tight or cotransitive apartness relations on nontrivial dcpos.

**Theorem 5.6** *Let $D$ be a dcpo with an apartness relation $\sharp$.*

(i) *If $D$ has elements $x \sqsubset y$, then tightness of $\sharp$ implies excluded middle.*

(ii) *If $D$ has elements $x \sqsubseteq y$ with $x \sharp y$, then cotransitivity of $\sharp$ implies weak excluded middle.*

**Proof.** (i): If $\sharp$ is tight, then $D$ is $\neg\neg$-separated by Lemma 5.2, which, since $D$ has elements $x \sqsubset y$, implies excluded middle by [10, Theorem 38].

(ii): For any proposition $P$, consider the supremum $s_P$ of the directed subset $\{x\} \cup \{y \mid P\}$. If $\sharp$ is cotransitive, then either $x \sharp s_P$ or $y \sharp s_P$. In the first case, $x \neq s_P$, so that $\neg\neg P$ must be the case. In the second case, $y \neq s_P$, so that $\neg P$ must be true. Hence, $\neg P$ is decidable and weak excluded middle follows. $\square$

**Theorem 5.7**

(i) *If excluded middle holds, then the intrinsic apartness on any dcpo is tight. In the other direction, if $D$ is a continuous dcpo with elements $x \sqsubseteq y$ that are intrinsically apart, then tightness of the intrinsic apartness on $D$ implies excluded middle.*

(ii) *If excluded middle holds, then the intrinsic apartness on any dcpo is cotransitive. In the other direction, if $D$ is a continuous dcpo and has elements $x \sqsubseteq y$ that are intrinsically apart, then cotransitivity of the intrinsic apartness on $D$ implies excluded middle.*

(iii) *In particular, if the intrinsic apartness on the Sierpiński domain $\mathbb{S}$ is tight or cotransitive, then excluded middle follows.*

We now isolate a collection of elements, which we call sharp elements, for which the intrinsic apartness *is* tight and cotransitive. The definition of a sharp element of a continuous dcpo may be somewhat opaque, but the algebraic case (Proposition 5.11) is easier to understand: an element $x$ is sharp if and only if $c \sqsubseteq x$ is decidable for every compact element $c$. Sharpness also occurs naturally in our examples in Section 7, e.g. a lower real is sharp if and only if it is located.

**Definition 5.8** An element $x$ of a dcpo $D$ is *sharp* if for every $y, z \in D$ with $y \ll z$ we have $y \ll x$ or $z \not\sqsubseteq x$.

Theorem 5.14 below provides many examples of sharp elements. Our first result is that sharpness is equivalent in continuous dcpos to a seemingly stronger condition.

**Proposition 5.9** *An element $x$ of a continuous dcpo $D$ is sharp if and only if for every $y, z \in D$ with $y \ll z$ we have $y \ll x$ or $z \not\sqsubseteq x$.*

**Lemma 5.10** *An element $x$ of a continuous dcpo $D$ with a basis $B$ is sharp if and only if for every $a, b \in B$ with $a \ll b$ we have $a \ll x$ or $b \not\sqsubseteq x$.*

**Proposition 5.11** *An element $x$ of an algebraic dcpo $D$ is sharp if and only if for every compact $c \in D$ it is decidable whether $c \sqsubseteq x$ holds.*

**Proposition 5.12** *Assuming excluded middle, every element of any dcpo is sharp. The sharp elements of the Sierpiński domain $\mathbb{S}$ are exactly $\bot$ and $\top$. Hence, if every element of $\mathbb{S}$ is sharp, then excluded middle follows.*

We can relate the notion of sharpness to Spitters' [35] and Kawai's [17, Definition 3.5] notion of a located subset: A subset $V$ of a poset $S$ is *located* if for every $s, t \in S$ with $s \ll t$, we have $t \in V$ or $s \notin V$.

**Proposition 5.13** *An element $x$ of a continuous dcpo is sharp if and only if the filter of Scott open neighbourhoods of $x$ is located in the poset of Scott opens of $D$.*

The following gives examples of sharp elements.

**Theorem 5.14** *Let $D$ be a continuous dcpo with a basis $B$.*

(i) *Assuming that $D$ is pointed, the least element of $D$ is sharp if $B$ satisfies ($\delta_\bot$).*

(ii) *Every element of $B$ is sharp if $B$ satisfies ($\delta_{\ll}$) or ($\delta_{\sqsubseteq}$). In particular, in these cases, the sharp elements are a Scott dense subset of $D$ in the sense of Proposition 2.16.*

*If $D$ is algebraic, then we can reverse the implications above for the basis of compact elements of $D$.*

(iii) *Assuming that $D$ is pointed, the least element of $D$ is sharp if and only if the set of compact elements of $D$ satisfies ($\delta_{\perp}$).*

(iv) *The compact elements of $D$ are sharp if and only if the set of compact elements of $D$ satisfies ($\delta_{\ll}$) or ($\delta_{\sqsubseteq}$).*

**Theorem 5.15**

(i) *If $y$ is a sharp element of a continuous $D$, then $\neg(x \not\sqsubseteq y)$ implies $x \sqsubseteq y$ for every $x \in D$. In particular, the intrinsic apartness on a continuous dcpo $D$ is tight on sharp elements.*

(ii) *The intrinsic apartness on a continuous dcpo $D$ is cotransitive with respect to sharp elements in the following sense: for every $x, y \in D$ and sharp element $z \in D$, we have $x \mathrel{\#} y \to (x \mathrel{\#} z \vee y \mathrel{\#} z)$.*

## 6 Strongly Maximal Elements

Smyth [34] explored the notion of a *constructively maximal* element, adapted from Martin-Löf's [23]. In an unpublished manuscript [14], Heckmann arrived at an equivalent notion, assuming excluded middle, as noted in [34, Section 8], and called it *strong maximality.* Whereas Smyth works directly with abstract bases and rounded ideal completions, we instead work with continuous dcpos. We use a simplification of Smyth's definition, but follow Heckmann's terminology. We show that every strongly maximal element is sharp, compare strong maximality to maximality highlighting connections to sharpness, and study the subspace of strongly maximal elements.

**Definition 6.1** Two points $x$ and $y$ of a dcpo $D$ are *Hausdorff separated* if we have disjoint Scott open neighbourhoods of $x$ and $y$ respectively.

**Definition 6.2** An element $x$ of a continuous dcpo $D$ is *strongly maximal* if for every $u, v \in D$ with $u \ll v$, we have $u \ll x$ or $v$ and $x$ are Hausdorff separated.

The following gives another source of examples of sharp elements.

**Proposition 6.3** *Every strongly maximal element of a continuous dcpo is sharp.*

**Corollary 6.4**

(i) *The intrinsic apartness on a continuous dcpo $D$ is tight on strongly maximal elements.*

(ii) *The intrinsic apartness on a continuous dcpo $D$ is cotransitive with respect to strongly maximal elements in the following sense: for every $x, y \in D$ and strongly maximal element $z \in D$, we have $x \mathrel{\#} y \to (x \mathrel{\#} z \vee y \mathrel{\#} z)$.*

**Lemma 6.5** *If a continuous dcpo $D$ has a basis $B$, then an element $x \in D$ is strongly maximal if and only if for every $a, b \in B$ with $a \ll b$, we have $a \ll x$ or $b$ and $x$ are Hausdorff separated.*

**Lemma 6.6** *An element $x$ of an algebraic dcpo $D$ is strongly maximal if and only if for every compact element $c \in D$ either $c \sqsubseteq x$ or $c$ and $x$ are Hausdorff separated.*

Smyth's formulation of strong maximality [34, Definition 4.1] (called *constructive maximality* there) is somewhat more involved than ours, but it is equivalent, as Proposition 6.10 shows.

**Definition 6.7** We say that two elements $x$ and $y$ of a dcpo $D$ can be *refined*, written $x \Uparrow y$, if there exists $z \in D$ with $x \ll z$ and $y \ll z$.

On [34, p. 362], refinement is denoted by $x \uparrow y$, but we prefer $x \Uparrow y$, because one might want to reserve $x \uparrow y$ for the weaker $\exists_{z \in D} ((x \sqsubseteq z) \wedge (y \sqsubseteq z))$.

**Lemma 6.8** *Two elements $x$ and $y$ of a continuous dcpo $D$ are Hausdorff separated if and only if there exist $a, b \in D$ with $a \ll x$ and $b \ll y$ such that $\neg(a \Uparrow b)$. Moreover, if $D$ has a basis $B$, then $x$ and $y$ are Hausdorff separated if and only if there exist such elements $a$ and $b$ in $B$.*

In [34, p. 362], the condition in Lemma 6.8 is taken as a definition (for basis elements) and such elements $a$ and $b$ are said to lie apart and this notion is denoted by $a \sharp b$. We can now translate Smyth's [34, Definition 4.1] from ideal completions of abstract bases to continuous dcpos.

**Definition 6.9** An element $x$ of a continuous dcpo $D$ is *Smyth maximal* if for every $u, v \in D$ with $u \ll v$, there exists $d \ll x$ such that $u \ll d$ or the condition in Lemma 6.8 holds for $v$ and $d$.

**Proposition 6.10** *An element $x$ of a continuous $D$ is strongly maximal if and only if $x$ is Smyth maximal.*

### *6.1 Maximality and Strong Maximality*

The name strongly maximal is justified by the following observation.

**Proposition 6.11 (cf. Proposition 4.2 in [34])** *Every strongly maximal element of a continuous dcpo is maximal.*

In the presence of excluded middle, [34, Corollary 4.4] tells us that the converse of the above is true if and only if the Lawson condition [34,19] holds for the dcpo (the Scott and Lawson topologies coincide on the subset of maximal elements.) Without excluded middle, the situation is subtler and involves sharpness, as we proceed to show.

**Lemma 6.12** *Suppose that we have elements $x$ and $y$ of a continuous dcpo $D$ such that*

(i) *$x$ and $y$ are both strongly maximal;*

(ii) *$x$ and $y$ have a greatest lower bound $x \sqcap y$ in $D$;*

(iii) *$x$ and $y$ are intrinsically apart.*

*Then, for any proposition $P$, the supremum $\bigsqcup S$ of the directed subset $S :\equiv \{x \sqcap y\} \cup \{x \mid \neg P\} \cup \{y \mid \neg\neg P\}$ is maximal, but $\bigsqcup S$ is sharp if and only if $\neg P$ is decidable. Hence, if $\bigsqcup S$ is strongly maximal, then $\neg P$ is decidable.*

**Proposition 6.13** *Let $P$ be the poset with exactly three elements $\bot \leq 0, 1$ and $0$ and $1$ unrelated. If every maximal element of the algebraic dcpo $\mathrm{Idl}(P)$ is strongly maximal, then weak excluded middle follows. In the other direction, if excluded middle holds, then every maximal elements of $\mathrm{Idl}(P)$ is strongly maximal.*

Theorem 4.3 of [34] states that an element $x$ of a continuous dcpo is strongly maximal if and only if every Lawson neighbourhood of $x$ contains a Scott neighbourhood of $x$. This requirement on neighbourhoods is, assuming excluded middle, equivalent to the Lawson condition (the Scott topology and the Lawson topology coincide on the subset of maximal elements), as used in [34, Corollary 4.4] and proved in [13, Lemma V-6.5]. Inspecting the proof in [13], we believe that excluded middle is essential. However, we can still prove a constructive analogue of [34, Theorem 4.3], but it requires a positive formulation of the subbasic opens of the Lawson topology, using the relation $\not\sqsubseteq\!\!\!\!/$ rather than $\not\sqsubseteq$.

**Definition 6.14** The subbasic *Lawson closed* subsets of a dcpo $D$ are the Scott closed subsets and the upper sets of the form $\uparrow x$ for $x \in D$. The subbasic *Lawson opens* are the Scott opens and the sets of the form $\{y \in D \mid x \not\sqsubseteq\!\!\!\!/ \; y\}$ for $x \in D$.

In the presence of excluded middle, the subset $\{y \in D \mid x \not\sqsubseteq\!\!\!\!/ \; y\}$ is equal to $D \setminus \uparrow x$, so with excluded middle the above definition is equivalent to the classical definition of the subbasic Lawson opens as in [13, pp. 211–212].

**Theorem 6.15 (cf. Theorem 4.3 of [34])** *An element $x$ of a continuous dcpo is strongly maximal if and only if $x$ is sharp and every Lawson neighbourhood of $x$ contains a Scott neighbourhood of $x$.*

By Proposition (5.12), every element is sharp if excluded middle is assumed, so in that case, we can drop the requirement that $x$ is sharp. Hence, in the presence of excluded middle, we recover Smyth's [34, Theorem 4.3] from Theorem 6.15.

### *6.2 The Subspace of Strongly Maximal Elements*

The classical interest in strong maximality comes from the fact that, while the subspace of maximal elements may fail to be Hausdorff [14, Example 4], the subspace of strongly maximal elements with the relative Scott topology is both Hausdorff and regular [34, Theorem 4.6]. We offer constructive proofs of these claims, with the proviso that the Hausdorff condition is formulated with respect to the intrinsic apartness.

**Proposition 6.16** *The subspace of strongly maximal elements of a continuous dcpo $D$ with the relative Scott topology is Hausdorff, i.e. if $x$ and $y$ are strongly maximal, then $x \mathrel{\#} y$ if and only if there are disjoint Scott opens $U$ and $V$ such that $x \in U$ and $y \in V$.*

**Proposition 6.17** *The subspace of strongly maximal elements of a continuous dcpo $D$ with the relative Scott topology is regular, i.e. every Scott neighbourhood of a point $x \in D$ contains a Scott closed neighbourhood of $x$.*

# 7 Examples

In this final section before the conclusion, we illustrate the foregoing notions of intrinsic apartness, sharpness and strong maximality, by studying several natural examples. The first three examples are generalized domain environments in the sense of Heckmann [14]: we consider dcpos $D$ and topological spaces $X$ such that $X$ *embeds* into the subspace of maximal elements of $D$. In fact, we will see that $X$ is homeomorphic to the subspace of *strongly* maximal elements of $D$. Specifically, we will consider Cantor space, Baire space and the real line. The final example shows that sharpness characterizes exactly those lower reals that are located.

## *7.1 The Cantor and Baire Domains*

Fix an inhabited set $A$ with decidable equality. Typically, we will be interested in $A = \mathbf{2} :\equiv \{0, 1\}$ and $A = \mathbb{N}$.

**Definition 7.1**

(i) Write $(A^*, \preceq)$ for the poset of finite sequences on $A$ ordered by prefix. We write $\mathcal{A}$ for the ideal completion of $(A^*, \preceq)$, which is an algebraic dcpo by Lemma 2.20.

(ii) For an infinite sequence $\alpha$, we write $\bar{\alpha}_n$ for the first $n$ elements of $\alpha$. Given a finite sequence $\sigma$, we write $\sigma \prec \alpha$ if $\sigma$ is an initial segment of $\alpha$.

(iii) We write $\mathbf{A}$ for the space $A^{\mathbb{N}}$ of infinite sequences on $A$ with the product topology, taking the discrete topologies on $\mathbb{N}$ and $A$. A basis of opens is given by the sets $\{\alpha \in \mathbf{A} \mid \sigma \prec \alpha\}$ for finite sequences $\sigma$. The space $\mathbf{A}$ has a natural notion of apartness: $\alpha \mathrel{\#_{\mathbf{A}}} \beta \iff \exists_{n\in\mathbb{N}}\, \bar{\alpha}_n \neq \bar{\beta}_n$.

**Definition 7.2** If we take $A = \mathbf{2}$, then $\mathbf{A}$ is Cantor space and we call $\mathcal{A}$ the *Cantor domain*; and if we take $A = \mathbb{N}$, then $\mathbf{A}$ is Baire space and we call $\mathcal{A}$ the *Baire domain*.

**Definition 7.3** We define an injection $\iota \colon \mathbf{A} \hookrightarrow \mathcal{A}$ by $\iota(\alpha) :\equiv \bigcup_{\sigma \prec \alpha} {\downarrow}\sigma = \{\tau \in A^* \mid \tau \prec \alpha\}$.

**Theorem 7.4** *The image of $\iota$ is exactly the subset of strongly maximal elements of $\mathcal{A}$.*

**Proposition 7.5** *Suppose that $A$ has at least two elements. If every maximal element of $\mathcal{A}$ is strongly maximal, then weak excluded middle holds.*

**Definition 7.6** *Markov's Principle* is the assertion that for every infinite binary sequence $\phi$ we have that $\neg(\forall_{n\in\mathbb{N}}\, \phi(n) = 0)$ implies $\exists_{k\in\mathbb{N}}\, \phi(k) = 1$.

Markov's Principle [6] follows from excluded middle, but is independent in constructive mathematics, i.e. Markov's Principle is not provable and neither is its negation.

**Theorem 7.7** *For the Cantor domain, the strongly maximal elements are exactly those elements that are both sharp and maximal. For the Baire domain, the strongly maximal elements are exactly the elements that are both sharp and maximal if and only if Markov's Principle holds.*

**Lemma 7.8** *The space $\mathbf{A}$ is $T_0$-separated with respect to $\#_{\mathbf{A}}$, i.e. for $\alpha, \beta \in \mathbf{A}$ we have $\alpha \mathrel{\#_{\mathbf{A}}} \beta$ if and only if there exists an open containing $x$ but not $y$ or vice versa.*

**Theorem 7.9** *The map $\iota$ is a homeomorphism from the space $\mathbf{A}$ of infinite sequences to the space of strongly maximal elements of the algebraic dcpo $\mathcal{A}$ with the relative Scott topology. Moreover, $\iota$ preserves and reflects apartness: $\alpha \mathrel{\#_{\mathbf{A}}} \beta$ if and only if $\iota(\alpha) \mathrel{\#_{\mathcal{A}}} \iota(\beta)$ for every two infinite sequences $\alpha, \beta \in \mathbf{A}$.*

## *7.2 Partial Dedekind Reals*

Recall the definition of a (two-sided) Dedekind real number.

**Definition 7.10** Given a pair $x = (L_x, U_x)$ of subsets of $\mathbb{Q}$, we write $p < x$ for $p \in L_x$ and $x < q$ for $q \in U_x$. A *Dedekind real* $x$ is a pair $(L_x, U_x)$ of subsets of $\mathbb{Q}$ satisfying the following properties:

(i) *boundedness*: there exist $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$ such that $p < x$ and $x < q$.

(ii) *roundedness*: for every $p, q \in \mathbb{Q}$, we have $p < x \iff \exists_{r\in\mathbb{Q}}(p < r) \wedge (r < x)$ and similarly, $x < q \iff \exists_{s\in\mathbb{Q}}(s < q) \wedge (x < s)$.

(iii) *transitivity*: for every $p, q \in \mathbb{Q}$, if $p < x$ and $x < q$, then $p < q$.

(iv) *locatedness*: for every $p, q \in \mathbb{Q}$ with $p < q$ we have $p < x$ or $x < q$.

**Definition 7.11** The topological space $\mathbb{R}$ is the set of all Dedekind real numbers whose basic opens are given by $\{x \in \mathbb{R} \mid p < x \text{ and } x < q\}$ for $p, q \in \mathbb{Q}$. The space $\mathbb{R}$ has a natural notion of apartness, namely: $x \mathrel{\#_{\mathbb{R}}} y \iff \exists_{p\in\mathbb{Q}}\, (x < p < y) \vee (y < p < x)$.

**Definition 7.12** Consider the set $\mathbb{Q} \times_< \mathbb{Q} :\equiv \{(p,q) \in \mathbb{Q} \times \mathbb{Q} \mid p < q\}$ ordered by defining $(p,q) \prec (r,s) \iff p < r < s < q$. The pair $(\mathbb{Q} \times_< \mathbb{Q}, \prec)$ is an abstract basis, so $\mathcal{R} :\equiv \mathrm{Idl}(\mathbb{Q} \times_< \mathbb{Q}, \prec)$ is a continuous dcpo and we refer to its elements as *partial Dedekind reals*.

**Lemma 7.13** *For every two rationals $p < q$ and $I \in \mathcal{R}$, we have $\downarrow(p,q) \ll I$ if and only if $(p,q) \in I$.*

**Definition 7.14** We define an injection $\iota \colon \mathbb{R} \hookrightarrow \mathcal{R}$ by $\iota(L_x, U_x) :\equiv \{(p,q) \mid p \in L_x, q \in U_x\}$. The map $\iota$ is well-defined precisely because a Dedekind real is required to be bounded, rounded and transitive.

**Theorem 7.15** *The image of $\iota$ is exactly the subset of strongly maximal elements of $\mathcal{R}$.*

With excluded middle, the image of $\iota$ is just the set of maximal elements of $\mathcal{R}$. The following result highlights the constructive strength of locatedness of Dedekind reals.

**Proposition 7.16** *If every maximal element of $\mathcal{R}$ is strongly maximal, then weak excluded middle holds.*

We conjecture that $\mathcal{R}$ is similar to the Baire domain in that the strongly maximal elements of $\mathcal{R}$ only coincide with the elements that are both sharp and maximal if a constructive taboo holds.

**Lemma 7.17** *The Dedekind real numbers are $T_0$-separated with respect to $\#_{\mathbb{R}}$, i.e. for $x, y \in \mathbb{R}$ we have $x \#_{\mathbb{R}} y$ if and only if there exists an open $U$ containing $x$ but not $y$ or vice versa.*

**Theorem 7.18** *The map $\iota$ is a homeomorphism from $\mathbb{R}$ to the space of strongly maximal elements of the continuous dcpo $\mathcal{R}$ with the relative Scott topology. Moreover, $\iota$ preserves and reflects apartness.*

*7.3 Lower Reals*

We now consider lower reals, which feature a nice illustration of sharpness.

**Definition 7.19** The pair $(\mathbb{Q}, <)$ is an abstract basis, so $\mathcal{L} :\equiv \mathrm{Idl}(\mathbb{Q}, <)$ is a continuous dcpo and we refer to its elements as *lower reals*.

**Lemma 7.20** *For every $p \in \mathbb{Q}$ and $L \in \mathcal{L}$, we have $\downarrow p \ll L$ if and only if $p \in L$.*

**Lemma 7.21** *If $L \in \mathcal{L}$ is a lower real, then the pair $(L, U)$ with $U :\equiv \left\{q \in \mathbb{Q} \mid \exists_{s \in \mathbb{Q} \setminus L}\, s < q\right\}$ is rounded and transitive in the sense of Definition 7.10. Moreover, if $\mathbb{Q} \setminus L$ is inhabited, then $(L, U)$ is bounded too.*

Classically, every lower real whose complement is inhabited determines a Dedekind real by the construction above. It is well-known that constructively a lower real may fail to be located. The following result offers a domain-theoretic explanation of that phenomenon.

**Theorem 7.22** *A lower real $L \in \mathcal{L}$ is sharp if and only if the pair $(L, U)$ with $U$ as in Lemma 7.21 is located.*

## 8 Conclusion

Working constructively, we studied continuous dcpos and the Scott topology and introduced notions of intrinsic apartness and sharp elements. We showed that our apartness relation is particularly well-suited for continuous dcpos that have a basis satisfying certain decidability conditions, which hold in examples of interest. For instance, for such continuous dcpos, the Bridges–Vîţă apartness topology and the Scott topology coincide. We proved that no apartness on a nontrivial dcpo can be cotransitive or tight unless (weak) excluded middle holds. But the intrinsic apartness is tight and cotransitive when restricted to sharp elements. If a continuous dcpos has a basis satisfying the previously mentioned decidability conditions, then every basis element is sharp. Another class of examples of sharp elements is given by the strongly maximal elements. In fact, strong maximality is closely connected to sharpness and the Lawson topology. For example, an element $x$ is strongly maximal if and only if $x$ is sharp and every Lawson neighbourhood of $x$ contains a Scott neighbourhood of $x$. Finally, we presented several natural examples of continuous dcpos that illustrated the intrinsic apartness, strong maximality and sharpness.

In future work, we would like to explore whether a constructive and predicative treatment is possible, in particular, in univalent foundations without Voevodsky's resizing axioms as in [9]. Steve Vickers also pointed out two directions for future research. The first is to consider formal ball domains [13, Example V-6.8], which may subsume the partial Dedekind reals example. The second is to explore the ramifications of Vickers' observation that refinability (Definition 6.7) is decidable, even when the order is not, if the dcpo is algebraic and 2/3 SFP [37, p. 157]. Related to the examples, there is still the question of whether we can derive a constructive taboo from the assumption that strong maximality of a partial Dedekind real follows from having both sharpness and maximality, as discussed right after Proposition 7.16. Finally, the Lawson topology deserves further investigation within a constructive framework.

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## Appendix

For lack of space, we have omitted a number of proofs. A version of this paper containing full proofs can be found here: `https://arxiv.org/pdf/2106.05064.pdf`. It also contains a section on an additional example, namely an alternative domain for Cantor space using exponentials and the lifting monad.