Monads on Categories of Relational Structures

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Motivation

- (Moggi, 1991): Monads *are* computational effects
  - ▷ categorical semantics via *Kleisli presentations*
  - ▷ (probabilistic) nondeterminism, exceptions, continuations, etc.
- (Plotkin/Power, 2001): effects via equations and operations
  - ▷ rather general account for presenting computational effects
  - ▷ computational effects *are* monads
  - ▷ (Linton, 1966): monads on Set = equational theories
- Recent syntactic-minded approaches to bases beyond Set:
  - ▷ (Adámek/Ford/Milius/Schröder, 2020):

    inequational theories = monads on Pos
  - ▷ (Mardare/Panangaden/Plotkin, 2016):

    quantitative algebraic theories (for monads on Met)

Overview

Core: universal algebra for monads on categories of relational structures

Contributions

1. **Presentations of monads** on model categories of infinitary Horn theories via relational theories
2. **Relational Logic**: sequent calculus for relational algebraic reasoning

Horn Theories and

Categories of Relational Structures

Categories of relational structures

Claim

Horn theories balance expressive power with ‘nice’ categorical structure.

- for instance, there are infinitary Horn theories for
  - Par: partial algebras and homomorphisms
  - Pos: partially ordered sets and monotone maps
  - Met: 1-bounded metric spaces and non-expansive maps
- Particulars: categories $\mathsf{Str}(\Pi, \mathcal{A})$ of Π-structures for
  - a finitary (single-sorted) relational signature Π
  - specified by a set $\mathcal{A}$ of infinitary Horn sentences:

$$\forall x.\ \bigwedge_{i \in I} \alpha_i(\bar{x}_i) \implies \beta(\bar{x}_\beta)$$

    where $\alpha_i \in \Pi$ and $\beta \in \Pi \sqcup \{=\}$.
  - Morphisms: relation-preserving maps

Horn theories

Horn theory for Pos

- signature: a single binary symbol $\leq$
- axioms:

$$\implies x \leq x \qquad \{x \leq y, y \leq z\} \implies x \leq y \qquad \{x \leq y, y \leq z\} \implies x = y$$

Unlike Pos, Met includes an infinitary axiom:

$$\{x =_{\epsilon'} y \mid \epsilon' > \epsilon\} \implies x =_{\epsilon} y \qquad \text{(Arch)}$$

Arity of a Horn theory

The Horn theory $(\Pi, \mathcal{A})$ is λ-*ary* if $\mathsf{card}\Phi < \lambda$ for all $\Phi \implies \psi \in \mathcal{A}$.

Key ingredient I: local presentability

Proposition

Given a λ-ary Horn theory $(\Pi, \mathcal{A})$, $\mathsf{Str}(\Pi, \mathcal{A})$ is a full reflective subcategory of $\mathsf{Str}(\Pi)$ closed under λ-directed colimits.

In particular:

- The inclusion $\mathsf{Str}(\Pi, \mathcal{A}) \hookrightarrow \mathsf{Str}(\Pi)$ has a left adjoint

$$\mathsf{Str}(\Pi) \xrightarrow{R} \mathsf{Str}(\Pi, \mathcal{A}) \qquad \text{(the } \textit{reflector}\text{)}$$

- $\mathsf{Str}(\Pi, \mathcal{A})$ is (co)complete and locally λ-presentable
    - X λ-presentable if $\mathsf{card} X < \lambda$ and X is λ-generated

Key ingredient II: closed structure

- Tensor: $\otimes\colon \mathsf{Str}(\Pi) \times \mathsf{Str}(\Pi) \to \mathsf{Str}(\Pi)$
  - ▷ **carrier**: the product $X_0 \times X_1$
  - ▷ **relations**: for $f\colon \mathsf{ar}(\alpha) \to X_0 \times X_1$,

$$X_0 \otimes X_1 \models \alpha(f) :\Longleftrightarrow \exists i \in \{0,1\}.\pi_i \cdot f \text{ is constant and } X_{i+1} \models \pi_{i+1} \cdot f$$

- Internal hom $[-,-]$ of $X, Y \in \mathsf{Str}(\Pi)$:
  - ▷ **carrier**: $\mathsf{Str}(\Pi)(X, Y)$
  - ▷ **relations**: point-wise structure on maps

Proposition

Let $(\Pi, \mathcal{A})$ be a λ-ary Horn theory. Then

$$(\mathsf{Str}(\Pi, \mathcal{A}), R \cdot \otimes, RI)$$

is locally λ-presentable as a symmetric monoidal closed category.

...so $[X, -]$ is λ-accessible for λ-presentable X

Presentations of Monads

on Categories of Horn Models

Algebras over Horn models

Assumption

$\mathscr{C} := \mathsf{Str}\mathscr{H}$ for a λ-ary Horn theory $\mathscr{H} = (\Pi, \mathcal{A})$, and $\kappa \leq \lambda$

- κ-ary *signature* Σ:
  - ▷ the arity of $\sigma \in \Sigma$, $\mathsf{ar}(\sigma)$, is an internally κ-presentable object
- We have a category of Σ-algebras, $\mathsf{Alg}\,\Sigma$:
  - ▷ **objects**: Σ-*algebras*
    a $\mathscr{C}$-object A equipped with $\mathscr{C}$-morphisms

$$\sigma_A \colon [\mathsf{ar}(\sigma), A] \to A \qquad (\sigma \in \Sigma)$$

  - ▷ **morphisms**: *homomorphisms*
    $\mathscr{C}$-morphism $A \to B$ making the following commute for all $\sigma \in \Sigma$:

$$\begin{array}{ccc} [\mathsf{ar}(\sigma), A] & \xrightarrow{\sigma_A} & A \\ {\scriptstyle h\cdot(-)}\downarrow & & \downarrow{\scriptstyle h} \\ [\mathsf{ar}(\sigma), B] & \xrightarrow{\sigma_B} & B \end{array}$$

Relational algebraic theories

κ-ary relational algebraic Σ-theory

Specified by a set $\mathcal{E}$ of Σ-*relations*: expressions $X \vdash \alpha(f)$ where

- X is a κ-presentable object
- $\alpha \in \Pi$, and
- f is a function $\mathsf{ar}(\alpha) \to T_\Sigma(X)$ (= Σ-terms over $|X|$, defined as usual)

Example: $\mathscr{C} = \mathsf{Pos}$

- Signature: a unary operation ξ
- Axiom:

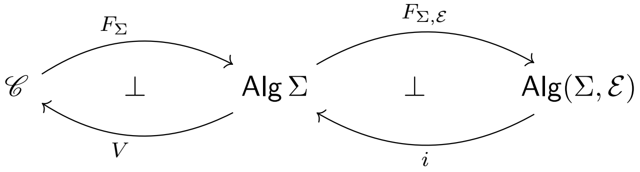
$$\{x\} \vdash x \leq \xi(x)$$

From theories to monads

Theorem

There is a translation of each κ-ary relational algebraic theory into a κ-accessible enriched monad on $\mathsf{Str}\mathscr{H}$, preserving categories of models.

- Proof idea:
  - ▷ Σ has a presentation as a κ-accessible functor
  - ▷ $\mathsf{Alg}(\Sigma, \mathcal{E})$ is a reflective subcategory of $\mathsf{Alg}\,\Sigma$
  - ▷ preservation of models: Beck's monadicity theorem



The ensuing monad is the *free-algebra monad* of $(\Sigma, \mathcal{E})$

From monads to theories

Monad-to-theory translation

Every λ-accessible monad $T\colon \mathsf{Str}\mathscr{H} \to \mathsf{Str}\mathscr{H}$ induces relational algebraic theory $\mathbb{T}$ described as follows:

- $\Sigma := \coprod_{\Gamma \in \mathscr{P}_\lambda} |T\Gamma|$
- $\mathbb{T}$ includes all axioms of the following shapes, where $\Gamma \in \mathscr{P}_\lambda$:
  1. $\Gamma \vdash \alpha(\sigma_i)$ for all $\sigma_i \in T\Gamma$ such that $T\Gamma \models \alpha(\sigma_i)$
  2. $\Gamma \vdash f^*(\sigma) = \sigma(f)$ for all $\sigma \in \Sigma$ and all morphisms $f\colon \mathsf{ar}(\sigma) \to T\Gamma$
  3. $\Gamma \vdash \eta_\Gamma(x) = x$ for all $x \in \Gamma$

$$f^* := TX \xrightarrow{Tf} TTY \xrightarrow{\mu_Y} TY \text{ for } f \in \mathscr{C}(X, TY)$$

Proposition

Each enriched λ-accessible monad T is the free-algebra monad of its associated relational algebraic theory.

Relational Logic and
a Construction of Free Algebras

Relational Logic

Sound/complete sequent calculus for relational reasoning:

$$X \vdash {\downarrow} t \quad (\text{“definedness”}) \qquad X \vdash \alpha(t_1, \ldots, t_{\mathsf{ar}(\alpha)}) \quad (\text{“relational”})$$

- “elimination rule for arity conditions” concludes definedness of operations:

$$(\mathsf{E\text{-}Ar})\ \frac{\{X \vdash \alpha(f \cdot g) \mid \mathsf{ar}(\sigma) \models \alpha(g)\} \cup \{X \vdash {\downarrow} f(i) \mid i \in \mathsf{ar}(\sigma)\}}{X \vdash {\downarrow} \sigma(f)}$$

  - ▷ map types: $\mathsf{ar}(\alpha) \xrightarrow{g} \mathsf{ar}(\sigma) \xrightarrow{f} T_\Sigma(X)$

- (general) substitution, cut, **subterm** and “**arity**” rules all admissible

Theorem

$X \vdash \alpha(f)$ is derivable iff every $A \in \mathsf{Alg}(\Sigma, \mathcal{E})$ satisfies $X \vdash \alpha(f)$.

Construction of Free Algebras

Construction of free $(\Sigma, \mathcal{E})$-algebras, briefly

For a $\mathscr{H}$-model X, the free $(\Sigma, \mathcal{E})$-algebra

- Step 1: form the Π-structure $\mathscr{T}_{\mathcal{E}}(X)$ with
  - ▷ **carrier**: terms $t \in T_\Sigma(X)$ such that $X \vdash \downarrow t$ derivable
  - ▷ **relations**: $\alpha(t_i) :\Longleftrightarrow X \vdash \alpha(t_i)$ is derivable
- Step 2: form the quotient of $\mathscr{T}_{\mathcal{E}}(X)$ by 'derivable equality'
  - ▷ this quotient admits the structure of a $\mathscr{H}$-model (!)

Theorem

For all $X \in \mathsf{Str}\mathscr{H}$, $\mathscr{T}_{\mathcal{E}}(X)$ carries the structure of a Σ-algebra with the universal property of a free $(\Sigma, \mathcal{E})$-algebra on X.

- In general, $\mathscr{T}(X)$ is **not** a quotient of $\mathscr{T}_{\mathcal{E}}(X)$
  - ▷ ...this is because (I-Ar) may create new defined terms

Concluding Remarks

Summary:

- For a λ-ary Horn theory $\mathscr{H}$, we have a bijective correspondence
  - λ-accessible enriched monads on $\mathsf{Str}\mathscr{H}$ and
  - λ-ary relational algebraic theories
- The theory-to-monad translation holds for all regular $\kappa \leq \lambda$
- Relational logic is sound/complete for relational reasoning

Future work:

- Generalization to the setting of graded monads
  - theory of 'behavioural relations' for Horn-definable relation types á la Milius, Pattinson, and Schröder (CALCO 2015)
- Further examples/enrichments?
- Which theories capture, e.g., finitary monads on Met?

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