The Central Valuations Monad

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Motivation

- Probability and recursion are important computational effects.
- *Domain Theory* – staple of denotational study of recursion.
- Adding probability to domain-theoretic approach has been difficult.
- Canonical approach: Kleisli category of the *valuations monad* $\mathcal{V}$ [1].
- Two major open problems unsolved since 1989.
- **Related work:** probabilistic coherence spaces, quasi-Borel spaces, cones, etc.
- **Recent work:** Three commutative submonads of $\mathcal{V}$, soundness and (strong) adequacy for *discrete* probabilistic choice [2].
- **This talk:** A commutative submonad of $\mathcal{V}$ for (continuous?) probabilistic choice.

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[1] Jones and Plotkin. *"A probabilistic powerdomain of evaluations."* **LICS 1989**.

[2] Jia, Lindenhovius, Mislove, Z. *"Commutative Monads for Probabilistic Programming Languages"* **LICS 2021**.

Background: Domain Theory (Dcpo's)

- Domain theory provides an order-theoretic view of computation and recursion.
- Two main classes of objects in domain theory: *dcpo's* and *domains*.
- A nonempty subset A of a *poset* D is *directed* if each pair of elements in A has an upper bound in A.
- A *directed-complete partial order* (dcpo) is a poset in which every directed subset A has a supremum $\sup A$.
  - **Example:** the unit interval $[0, 1]$ is a dcpo in the usual ordering.
  - **Example:** the open sets of a topological space in the inclusion order.
- A function $f : D \to E$ between two dcpo's is *Scott-continuous* if it is monotone and preserves suprema of directed subsets.
- The category **DCPO** of dcpo's and Scott-continuous functions is *cartesian closed*, complete and cocomplete.
- The category **DCPO** is very important for denotational semantics.

Background: Domain Theory (Domains)

- A *domain*, also known as a *continuous* dcpo, is a dcpo equipped with a notion of approximation (details omitted).
- Domains may be thought of as very well-behaved dcpo's.
- The category of domains and Scott-continuous maps is denoted by **DOM**.
- **Problem:** The category **DOM** is *not* cartesian closed.

Background: Domain Theory (Scott Topology)

- The order on a dcpo X induces a canonical topology σX, called the *Scott-topology*.
- The *Scott topology* σD on a dcpo D consists of the upper subsets $U = \uparrow U = \{x \in D \mid \exists u \in U.\ u \leq x\}$ that are *inaccessible by directed suprema:* i.e., if $A \subseteq D$ is directed and $\sup A \in U$, then $A \cap U \neq \emptyset$.
- The topological space $(D, \sigma D)$ is also written as ΣD.
- $f : X \to Y$ is Scott-continuous iff f is continuous w.r.t. ΣX and ΣY.

Background: Probability and Recursion

- How to talk about recursion *and* probability?
- Why not just take $\mathrm{Meas}(X)$, the set of subprobability measures on the Borel σ-algebra induced by the Scott-topology of a dcpo X?
- Because it is unclear how to extend the assignment $\mathrm{Meas}(-)$ to a monad over **DCPO**.
- A monadic semantics over **DCPO** seems very unlikely with this approach.

Background: Valuations

- The domain-theoretic approach to probability is based on valuations [1].
- A *subprobability valuation* on a dcpo X is a Scott-continuous map $\nu : \sigma X \to [0,1]$, which is strict ($\nu(\emptyset) = 0$) and modular ($\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$).
  - **Example:** The always-zero valuation $\mathbf{0}$.
  - **Example:** For $x \in X$, δ_x is defined as $\delta_x(U) = 1$ if $x \in U$ and $\delta_x(U) = 0$ otherwise.
- The set of subprobability valuations on a dcpo X, denoted $\mathcal{V}X$, is a *pointed dcpo* in the stochastic order: $\nu_1 \leq \nu_2$ iff $\forall U \in \sigma X . \nu_1(U) \leq \nu_2(U)$.
- **Remark:** Valuations are similar to Borel measures and in some cases coincide.

[1] Jones and Plotkin. *"A probabilistic powerdomain of evaluations."* **LICS 1989**.

Background: Valuations Monad

- The assignment $\mathcal{V}(-)$ can be equipped with the structure of a *strong monad*.
- Given $h : D \to E$, define $\mathcal{V}(h) : \mathcal{V}D \to \mathcal{V}E :: \nu \mapsto \lambda U.\nu(h^{-1}(U))$.
- The unit of $\mathcal{V}$ is given by $\eta_D : D \to \mathcal{V}D :: x \mapsto \delta_x$.
- A notion of integration can be defined. Given $\nu \in \mathcal{V}X$ and $f : X \to [0,1]$ Scott-continuous, we can define the *integral of f against* ν by:

$$\int_{x\in X} f(x) d\nu \stackrel{\text{def}}{=} \int_0^1 \nu(f^{-1}((t,1]))dt.$$

- The multiplication is given by $\mu_D : \mathcal{V}\mathcal{V}D \to \mathcal{V}D :: \varpi \mapsto \lambda U. \int_{\nu\in\mathcal{V}D} \nu(U) d\varpi$.
- The strength is $\tau_{DE} : D \times \mathcal{V}E \to \mathcal{V}(D \times E) :: (x,\nu) \mapsto \lambda U. \int_{y\in E} \chi_U(x,y) d\nu$.

Background: Problems of the Valuations Monad

- The monad $\mathcal{V}$ is *strong* on **DCPO** and *commutative* on **DOM** [3].
- Two major open problems since 1989:
  - **Problem:** Is $\mathcal{V}$ a commutative monad on **DCPO**?
  - **Problem (Jung-Tix):** Find a cartesian closed category of *domains* on which $\mathcal{V}$ is a commutative monad.
- Having a domain-theoretic model with a *commutative valuations monad* over a *cartesian closed category* is important for the semantics. Do they exist?
- Yes [2]. We use *topological methods* to construct *commutative submonads* of $\mathcal{V}$.
  - We have shown our monads are suitable for *discrete* probabilistic choice.

[2] Jia, Lindenhovius, Mislove, Z. *"Commutative Monads for Probabilistic Programming Languages"* **LICS 2021**.

[3] Jones. *Probabilistic non-determinism.* **PhD Thesis, University of Edinburgh, 1990.**

Background: Discrete vs Continuous Probabilistic Choice

- A programming language with discrete probabilistic choice:
  - A term M can reduce to *countably* many values.
  - $P(M \to_* V)$ is the probability that term M reduces to value V (operational notion).
  - In the denotational semantics, *strong adequacy* is the statement:

$$[\![M]\!] = \sum_{V \in \mathrm{Val}(M)} P(M \to_* V)[\![V]\!] \tag{1.1}$$

- A programming language with continuous probabilistic choice:
  - A term M can reduce to *uncountably* many values.
  - $P(M \to_* -)$ is a *subprobability measure* determined by the operational semantics.
  - In the denotational semantics, *strong adequacy* is the statement:

$$[\![M]\!] = \int_{V \in \mathrm{Val}(M)} [\![V]\!] d\ P(M \to_* V) \tag{1.2}$$

- Our LICS'21 monads are strongly adequate for discrete probabilistic choice.

Fubini $\iff$ Commutativity of $\mathcal{V}$

- Commutativity of the monad $\mathcal{V}$ is equivalent to showing the Fubini-style equation

$$\int_{x\in D}\int_{y\in E}\chi_U(x,y)d\xi d\nu = \int_{y\in E}\int_{x\in D}\chi_U(x,y)d\nu d\xi$$

  for dcpo's D and E, for $U \in \sigma(D\times E)$ and for $\nu \in \mathcal{V}D, \xi \in \mathcal{V}E$.
- This equation is known to hold if D or E is a domain.
  - This is why $\mathcal{V}$ is commutative on **DOM**.
- This equation is known to hold if ν or ξ is a *point-continuous* valuation.
  - This is why our monad $\mathcal{P}$ (LICS'21) is a commutative submonad of $\mathcal{V}$.
- We use *topological methods* to define and show that our LICS'21 monads are commutative submonads of $\mathcal{V}$.

The Central Valuations Monad

- The main idea behind our new commutative submonad is *algebraic*.
  - Recall that the *centre* of a group is always an abelian subgroup.
  - Recall that the *centre* of a premonoidal category is always a monoidal subcategory.
- **Definition:** A subprobability valuation ν on a dcpo D is called a *central valuation* if for any dcpo E, any valuation μ on E, and any Scott-continuous function $h\colon D \times E \to [0,1]$, we have

$$\int_{x \in D} \int_{y \in E} h(x,y) d\mu d\nu = \int_{y \in E} \int_{x \in D} h(x,y) d\nu d\mu.$$

- We write $\mathcal{Z}D$ for the set of all central valuations on a dcpo D.
- **Theorem:** The assignment $\mathcal{Z}(-)$ extends to a commutative monad over the category **DCPO** when equipped with the (co)restricted monad operations of $\mathcal{V}$. In other words, $\mathcal{Z}$ is a commutative submonad of $\mathcal{V}$.

How large is the Central Valuations Monad?

- All of our monads from [2] are submonads of $\mathcal{Z}$. For every dcpo D :

$$\mathcal{S}D \subseteq \mathcal{M}D \subseteq \mathcal{W}D \subseteq \mathcal{P}D \subseteq \mathcal{Z}D \subseteq \mathcal{V}D.$$

- $\mathcal{Z}$ is large enough for *discrete* probabilistic choice.
- $\mathcal{Z}$ is the largest commutative submonad of $\mathcal{V}$ known so far.
- $\mathcal{Z} = \mathcal{V}$ iff $\mathcal{V}$ is commutative on **DCPO** (open problem for 32 years).
- **Theorem:** Let $f : [0,1] \to D$ be a lower semi-continuous map into a dcpo D. If ν is any valuation on $[0,1]$, then $f_*(\nu) \stackrel{\text{def}}{=} \lambda O \in \sigma D.\nu(f^{-1}(O))$ is in $\mathcal{Z}D$.
  - We have not been able to prove this theorem for $\mathcal{M}, \mathcal{W}$ or $\mathcal{P}$.
- **Work-in-progress:** Is $\mathcal{Z}$ large enough for *continuous* probabilistic choice?

[2] Jia, Lindenhovius, Mislove, Z. **LICS 2021**.

Thank you for your attention!