Coderelictions for Free Exponential Modalities
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Background

- In categorical models of linear logic, the exponential modality $!$ is interpreted as a **monoidal coalgebra modality**, which in particular is a symmetric monoidal comonad $!$ such that for every object A, $!A$ is a cocommutative comonoid.

- An important source of examples of monoidal coalgebra modalities are the **free exponential modalities**, where $!A$ is also the cofree cocommutative comonoid over A. Categorical models of linear logic with free exponential modalities are known as **Lafont categories**.

- Differential linear logic is an extension of linear logic which includes a differentiation inference rule. Categorical models of differential linear logic are called **differential categories**, where the extra differential structure this can be equivalently interpreted either as a **deriving transformation** $\mathsf{d}_A : !A \otimes A \to !A$ or a **codereliction** $\eta_A : A \to !A$.

- There are many examples of differential categories whose $!$ is a free exponential modality. This raises the question of if free exponential modalities (in an appropriate setting) always comes equipped with a codereliction/deriving transformation, and if a Lafont category is always a differential category...

Today's Story

- The answer is **YES**! In fact, in this paper:

  R. Blute, K. O'Neil, R. Lucyshyn-Wright **Derivations in Codifferential Categories.** (2016)

  It was shown that every free exponential modality admits a deriving transformation, and thus every Lafont category with finite biproducts is a differential category.

- However, from a differential linear logic perspective, this approach is not the usual one since:

  1. The result was stated in the dual setting;
  2. The proof and construction involve the deriving transformation rather than the codereliction.

  The latter reason is important since in differential linear logic, it is often the codereliction η that is preferred and plays a more central role instead of the deriving transformation d.

- **TODAY'S STORY:** provide an alternative proof that every Lafont category is a differential category by showing that every free exponential modality comes equipped with a unique codereliction η, which we will construct using the couniversal property of $!A$, and where the necessary axioms are satisfied almost automatically simply by construction

Summary

- Review monoidal coalgebra modalities and their !-coalgebras.
- Review free exponential modalities.
- Review coderelictions and differential categories.
- **NEW**: Introduce **infinitesimal augmentations**, a new axiomatization of differential categories in terms of !-coalgebras.
- Explain how infinitesimal augmentations and coderelictions are the same.
- Provide an alternative proof that free exponential modalities have a codereliction by easily constructing its infinitesimal augmentation.

Monoidal Coalgebra Modalities

Let $\mathbb{X}$ be a symmetric (strict) monoidal category (with tensor $\otimes$ and unit k) and with finite products (with binary product $\times$ and terminal object $\top$).

Definition

A **monoidal coalgebra modality** on $\mathbb{X}$ consists of:

- A comonad $(!, \delta, \varepsilon)$ which recall consists of:

$$! : \mathbb{X} \to \mathbb{X} \qquad \delta_A : !A \to !!A \qquad \varepsilon_A : !A \to A$$

  satisfying certain identities.

- Equipped with two natural transformations Δ and e:

$$\Delta_A : !A \to !A \otimes !A \qquad \mathsf{e}_A : !A \to k$$

  such that $(!A, \Delta_A, \mathsf{e}_A)$ is a cocommutative comonoid and δ_A is a comonoid morphism.

- Equipped with two natural transformations m and m_k:

$$\mathsf{m}_{A,B} : !A \otimes !B \to !(A \otimes B) \qquad \mathsf{m}_k : k \to !k$$

  such that ! is a symmetric monoidal comonad + extra coherences with Δ and e.

One then has the **Seely isomorphisms** that: $!(\top) \cong k$ and $!(A \times B) \cong !A \otimes !B$.

!-Coalgebras

Recall that for a $(!, \delta, \varepsilon)$:

- A !-coalgebra is a pair $(A, \omega : A \to !A)$ such that:

$$\begin{array}{ccc} A & \xrightarrow{\omega} & !A \\ & \searrow^{=} & \downarrow{\scriptstyle \varepsilon_A} \\ & & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\omega} & !A \\ {\scriptstyle \omega}\downarrow & & \downarrow{\scriptstyle \delta_A} \\ !A & \xrightarrow[\omega]{} & !!A \end{array}$$

- The !-coalgebra $(!A, \delta_A)$ is called the cofree !-coalgebra over A.
- The category of !-coalgebras and !-coalgebra morphisms is called the **coEilenberg-Moore category** of the comonad $(!, \delta, \varepsilon)$ and will be denoted $\mathbb{X}^!$.

Proposition

The coEilenberg-Moore category of a monoidal coalgebra modality has finite products where (k, m_k) is a terminal object and

$$(A, \omega) \otimes (B, \nu) := (A \otimes B,\ A \otimes B \xrightarrow{\omega \otimes \nu} !A \otimes !B \xrightarrow{\mathsf{m}_{A,B}} !(A \otimes B)\ )$$

So $\otimes$ of the base category becomes a product in the coEilenberg-Moore category.

If (A, ω) is a !-coalgebra, then the triple $(A, \Delta^\omega, \mathsf{e}^\omega)$ is a cocommutative comonoid where:

$$\Delta^\omega := \ A \xrightarrow{\omega} !A \xrightarrow{\Delta_A} !A \otimes !A \xrightarrow{\varepsilon_A \otimes \varepsilon_A} A \otimes A \qquad \mathsf{e}^\omega := \ A \xrightarrow{\omega} !A \xrightarrow{\mathsf{e}_A} k$$

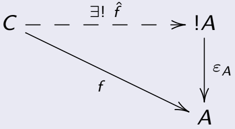
So every !-coalgebra is also a cocommutative comonoid.

Free Exponential Modalities

Definition

A **free exponential modality** is a monoidal coalgebra modality[a] ! where $!A$ is the cofree cocommutative comonoid over A, that is:

- if (C, Δ, e) is a cocommutative comonoid, then for every map $f : C \to A$ there exists a unique comonoid morphism $\hat{f} : (C, \Delta, \mathrm{e}) \to (!A, \Delta, \mathrm{e})$ such that:



A **Lafont category** is a symmetric monoidal category with finite products and a free exponential modality.

[a] It is possible to define free exponential modalities from scratch and show that they are always monoidal coalgebra modalities.

The !-coalgebras of a free exponential modality are precisely the cocommutative comonoids, in other words: the coEilenberg-Moore category of a free exponential modality is equivalent to the category of cocommutative comonoids.

Examples of Free Exponential Modalities

Example

Let REL be the category of sets and relations. Then REL has a free exponential modality where for a set X, $!X = \{[\![x_1, \ldots, x_n]\!] \mid x_i \in X\}$ is the set of finite multisets/bags of X.

Example

Let k be a field and let VEC_k be the category of k-vector spaces and k-linear maps between them. Then VEC_k^{op} has a free exponential modality given by the free symmetric algebra, $!V = \mathrm{Sym}(V)$, which is the free commutative k-algebra. In particular, if X is a basis for V, then $!V \cong k[X]$.

Example

VEC_k has a free exponential modality given where $!V$ is the cofree cocommutative k-coalgebra over V. When k is a field of characteristic 0 and if X is a basis of V, then $!V \cong \bigoplus_{v \in V} k[X]$.

Clift, J. and Murfet, D., **Cofree coalgebras and differential linear logic**.

For a list of more examples, see:

M. Hyland, A. Schalk, **Glueing and orthogonality for models of linear logic**

P-A. Melliès, N. Tabareau, C. Tasson, **An explicit formula for the free exponential modality of linear logic.**

Additive Structure and Biproducts

Before we switch gears to talk about differential stuff, we must first discuss additive structure:

Definition

An additive symmetric monoidal category is a symmetric monoidal category $\mathbb{X}$ which is enriched over commutative monoids, that is, every hom-set is a commutative monoid with an addition operation $+$ and a zero 0, such that the additive structure is preserves by composition and $\otimes$:

$$f \circ 0 \circ g = 0 \qquad f \circ (g + h) \circ k = f \circ g \circ k + f \circ h \circ k$$

$$f \otimes 0 \otimes g = 0 \qquad f \otimes (g + h) \otimes k = f \otimes g \otimes k + f \otimes h \otimes k$$

If an additive symmetric monoidal category has finite products, then said finite product structure is in fact a finite biproduct structure that is distributive. We denote the zero object as 0, and the biproduct as $\oplus$ and we have that:

$$(A \oplus B) \otimes (C \oplus D) \cong (A \otimes C) \oplus (A \otimes D) \oplus (B \otimes C) \oplus (B \otimes D) \qquad A \otimes 0 \cong 0 \cong 0 \otimes A$$

If $!$ is a monoidal coalgebra modality on an additive symmetric monoidal category, then we have:

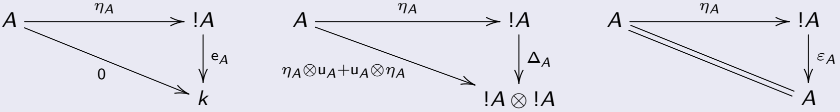
$$\nabla_A : !A \otimes !A \to !A \qquad \mathsf{u}_A : k \to !A$$

which makes $!A$ into a bialgebra, so in particular a commutative monoid.

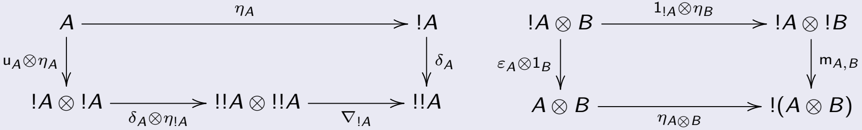
Definition

For a monoidal coalgebra modality ! on an additive symmetric monoidal category (with finite biproducts), a **codereliction** is a natural transformation $A \xrightarrow{\eta_A} !A$ such that:

[dC.1] Constant Rule: **[dC.2]** Product Rule: **[dC.3]** Linear Rule:



[dC.4′] Alternative Chain Rule: **[dC.m]** Monoidal Rule:



A **differential (storage) category** is an additive symmetric monoidal category with finite biproducts with a monoidal coalgebra modality ! that comes equipped with a codereliction η.

A quick word on deriving transformations

Alternatively, a differential category can defined in terms of a deriving transformation $\mathsf{d}_A : !A \otimes A \to !A$ satisfying five axioms (such as the product rule and the chain rule).

Proposition

For monoidal coalgebra modalities, deriving transformation are in bijective correspondence with coderelictions.

$$\eta := \quad A \xrightarrow{\mathsf{u}_A \otimes 1_A} !A \otimes A \xrightarrow{\mathsf{d}_A} !A$$

$$\mathsf{d} := \quad !A \otimes A \xrightarrow{1_{!(A)} \otimes \eta_A} !A \otimes !A \xrightarrow{\nabla_A} !A$$

M. Fiore **Differential Structure in Models of Multiplicative Biadditive Intuitionistic Linear Logic**

R. Blute, R. Cockett, R.A.G. Seely, J-S. P. Lemay **Differential Categories Revisited**

Examples of Differential Categories

Example

In REL:

$$d_X := \{(([\![x_1, \dots, x_n]\!], x), [\![x, x_1, \dots, x_n]\!]) \mid x, x_i \in X\} \subseteq (!X \times X) \times !X$$

$$\eta_X := \{(x, [\![x]\!]) \mid x \in X\} \subseteq X \times !X$$

Example

In VEC_k^{op}, if X is a basis of V:

$$d_V : k[X] \to k[X] \otimes V \qquad d\left(p(x_1, \dots, x_n)\right) = \sum_{i=1}^{n} \frac{\partial p}{\partial x_i}(x_1, \dots, x_n) \otimes x_i$$

$$\eta_V : k[X] \to V \qquad \eta\left(\sum a_j x_1^{j_1} \dots x_n^{j_n}\right) = \sum a_j x_i$$

Example

In VEC_k, if X is a basis of V,

$$d_V : \left(\bigoplus_{v \in V} k[X]\right) \otimes V \to \bigoplus_{v \in V} k[X] \qquad d_V(p_v(x_1, \dots, x_n) \otimes x_i) = p_v(x_1, \dots, x_n) x_i$$

$$\eta_V : V \to \bigoplus_{v \in V} k[X] \qquad \eta(v) = \eta(\sum a_i x_i) = (\sum a_i x_i)_0$$

Coderelictions for Free Exponential Modalities

Proposition

Every additive Lafont category is a differential category, that is, every free exponential modality has a deriving transformation.

R. Blute, K. O'Neil, R. Lucyshyn-Wright **Derivations in Codifferential Categories.**

I will now give an alternative proof in terms of the codereliction...

To do this: provide alternative axiomatization of differential categories using !-coalgebras!

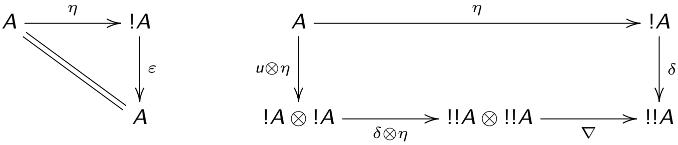
Coderelictions vs. !-coalgebras

Consider the type of the codereliction:

$$\eta_A : A \to !A$$

it may be tempting to think that this equips A with a !-coalgebra structure...

But this doesn't work!



The diagram on the right is the problem....

Another way of seeing this is because A may not have a comonoid structure, in particular it may not have a counit! Indeed, $0 : A \to A \otimes A$ works as a comultiplication but we don't necessarily have a map $A \to k$ which satisfies the necessary counit identities.

Coderelictions vs. !-coalgebras

Ok so what do we do? The answer is that we take a trick out of:

P-A. Melliès, N. Tabareau, C. Tasson, **An explicit formula for the free exponential modality of linear logic.**

Suppose that we have bipoducts $\oplus$, with injections $\iota_j : A_j \to A_0 \otimes A_1$ and projections $\pi_j : A_0 \otimes A_1 \to A_j$.

Then for every object A, it turns out that $k \oplus A$ has a canonical comonoid structure:

$$\Lambda_A := \; k \oplus A \xrightarrow{(\iota_0 \otimes \iota_0)\circ\pi_0 + (\iota_1 \otimes \iota_0)\circ\pi_1 + (\iota_0 \otimes \iota_1)\circ\pi_1} (k \oplus A) \otimes (k \oplus A)$$

$$k \oplus A \xrightarrow{\pi_0} k$$

Using element notation:

$$\Lambda_A(r, a) = (r, 0) \otimes (1, 0) + (0, a) \otimes (1, 0) + (1, 0) \otimes (0, a) \qquad e(r, a) = r$$

So there's a $k \otimes k$, $A \otimes k$ and $k \otimes A$ part, but no $A \otimes A$ part.

IDEA: Every codereliction induces a !-coalgebra structure on $k \oplus A$.

Infinitesimal Augmentations

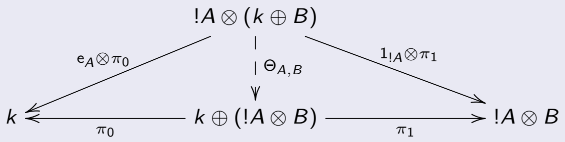
Definition

For a monoidal coalgebra modality ! on an additive symmetric monoidal category with finite biproducts, an **infinitesimal augmentation** is a natural transformation $\mathsf{H}_A : k \oplus A \to !(k \oplus A)$ such that:

[IA.1] $(k \oplus A, \mathsf{H}_A)$ is an !-coalgebra;

[IA.2] $\mathsf{H}_A : (k \oplus A, \Lambda_A, \pi_0) \to (!(k \oplus A), \Delta_{k \oplus A}, \mathsf{e}_{k \oplus A})$ is a comonoid morphism;

[IA.3] The canonical strength map $\Theta_{B,A} : !B \otimes (k \oplus A) \to k \oplus (!B \otimes A)$ defined as:



Then $(!A, \delta_A) \otimes (k \oplus B, \mathsf{H}_B) \xrightarrow{\Theta_{A,B}} (k \oplus (!A \otimes B), \mathsf{H}_{!A \otimes B})$ is a !-coalgebra morphism.

The canonical comonoid structure of $(k \oplus A, \mathsf{H}_A)$ is precisely that from the previous slide, that is:

$$\Delta^{\mathsf{H}_A} = \Lambda_A \qquad\qquad \mathsf{e}^{\mathsf{H}_A} = \pi_0$$

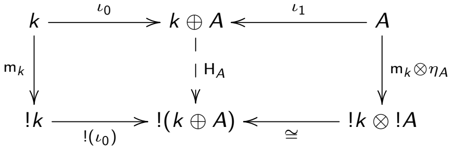
Proposition

There is a bijective correspondence between coderelictions and infinitesimal augmentations.

Proof: The axioms of infinitesimal augmentation are analogous to those of a codereliction:

$$[\mathbf{IA}.\mathbf{1}] = [\mathbf{dc}.\mathbf{3}] + [\mathbf{dc}.\mathbf{4}] \qquad [\mathbf{IA}.\mathbf{2}] = [\mathbf{dc}.\mathbf{1}] + [\mathbf{dc}.\mathbf{2}] \qquad [\mathbf{IA}.\mathbf{3}] = [\mathbf{dc}.\mathbf{5}]$$

$\eta \Rightarrow \mathsf{H}$: Define H as the unique map:



$H \Rightarrow \eta$: Define η as follows:

$$\eta_A := A \xrightarrow{\iota_1} k \oplus A \xrightarrow{\mathsf{H}_A} !(k \oplus A) \xrightarrow{!(\pi_1)} !A$$

Coderelictions for Free Exponential Modalities

Proposition

Free exponential modalities have a unique infinitesimal augmentation, and therefore have a unique codereliction.

Proof: Recall that !-coalgebras for free exponential modalities are precisely cocommutative comonoids. So since $k \oplus A$ is a cocommutative comonoid, it has a !-coalgebra structure $(k \oplus A, \mathsf{H}_A)$. Explicitly, H_A is defined as follows:

$$\begin{array}{ccc} k \oplus A & \overset{\exists!\ \mathsf{H}_A}{\dashrightarrow} & !(k \oplus A) \\ & \searrow^{=} & \downarrow \varepsilon_{k \oplus A} \\ & & k \oplus A \end{array}$$

From here, it remains only to check the strength rule, which amounts to asking that Θ is a comonoid morphism (which it always is!). Uniqueness follows from the universal property of !. As a result, we also obtain a codereliction, which is also unique. ■

Alternatively, $\eta : A \rightarrow !A$ can be built as follows:

$$\eta_A = \quad A \xrightarrow{\iota_1} k \oplus A \xrightarrow{\pi_1^\flat} !A \qquad\qquad \begin{array}{ccc} k \oplus A & \overset{\exists!\ \pi_1^\flat}{\dashrightarrow} & !A \\ & \searrow_{\pi_1} & \downarrow \varepsilon_A \\ & & A \end{array}$$

Terminology

A quick word on the terminology behind the name “infinitesimal augmentation”.

- “Augmentation” is a reference to the fact that $k \oplus A$ is always an augmented (co)algebra in the classical sense, in particular since $k \oplus A$ is the (co)free (co)pointed object over A.

- “Infinitesimal” is related to tangent category terminology.

  R. Cockett, G. Cruttwell. **Differential structure, tangent structure, and SDG**

- A tangent category is a category equipped with an endofunctor T and various other natural transformations whose axioms generalize the theory of smooth manifolds and their tangent bundles. A representable tangent category is a tangent category such that $\mathsf{T} \cong (-)^D$, where D is called an **infinitesimal object**.

- In this paper:

  R. Cockett, J-S. P. Lemay, R. Lucyshyn-Wright **Tangent Categories from the Coalgebras of Differential Categories.**

  It was explained how the coEilenberg-Moore category of a differential category is a representable tangent category. It turns out that the infinitesimal object is $(k \oplus k, \mathsf{H}_k)$!

- In future work, it would be interesting to further study the connection between infinitesimal augmentations and tangent structure. In particular, infinitesimal augmentations may provide the key in generalizing linear-non-linear adjunctions for differential categories...

Conclusion

- It is always of mathematical interest to have different proofs of the same result, especially when said proofs take different approaches.
- In this case, the proof given here has a more differential linear logic flavour to it, and is mostly automatic using the universal property of free exponential modalities.
- We hope that this paper will help open the door to revisiting other examples of Lafont categories and studying them from a differential category point of view.
- A possible advantage of infinitesimal augmentations is that the notions of !-coalgebras and comonoid morphisms are well-known, even to those who are not familiar with differential categories, and provide yet another way of understanding differentiation.
- Infinitesimal augmentations are closely linked to the notion of tangent categories. This should be studied more!

HOPE YOU ENJOYED MY TALK!

THANKS FOR LISTENING!

MERCI!