Initial Algebras Without Iteration

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Initial Algebras With Iteration

Initial Algebra Theorem (Trnková, Adámek, Koubek, Reiterman'75)

For a functor $F\colon \mathsf{Set} \to \mathsf{Set}$ tfae:

more generally: category $\mathcal{C}$ with *well-behaved* class $\mathcal{M}$ of monos; F preserving $\mathcal{M}$

1. an initial algebra exists,
2. a fixed point exists, (X with $FX \cong X$)
3. a pre-fixed point exists, ($FX \overset{m}{\rightarrowtail} X$ monic)
4. the initial-algebra chain converges.

Initial-algebra chain: the unique ordinal-indexed chain

$$0 \to F0 \to FF0 \to \cdots \to W_\omega \to W_{\omega+1} \to \cdots \to W_i \xrightarrow{w_{i,i+1}} W_{i+1} \to \cdots$$

$W_\omega = \operatorname{colim} F^i 0$, $W_{\omega+1} = FW_\omega$, $W_{i+1} = FW_i$

The chain converges if $w_{i,i+1}$ is an iso for some ordinal i.

Initial Algebras Without Iteration

Theorem

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Our contribution: A new compact and constructive proof based on

- Pataraia's fixed point theorem for dcpos with $\bot$
- pre-fixed points having smooth subobjects
- properties of recursive coalgebras

Ingredient 1: Pataraia's Theorem

Pataraia's and other Fixed Point Theorems

Kleene — Every continuous function on an ω-cpo with $\bot$ has a least fixed point.

Zermelo — Every monotone function on a chain-complete poset has a least fixed point.

(chain-complete poset $=$ dcpo with $\bot$ (Markowsky 1976))

Knaster-Tarski — Every monotone function on a complete lattice has least and greatest fixed point.

Theorem (Pataraia 1997)

Let P be a dcpo with $\bot$.

Then every monotone map $f : P \to P$ has a least fixed point μf.

Pataraia Induction Principle: Suppose $S \subseteq P$ fulfils

1. $\bot \in S$,
2. $s \in S \implies f(s) \in S$, ($S$ closed under f)
3. $\bigvee D \in S$ for $D \subseteq S$ directed. (S closed under directed joins)

Then $\mu f \in S$.

Ingredient 2:
Smooth of Subobjects and Monos

Smoothness

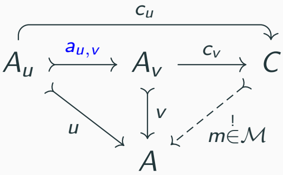
Definition

Let $\mathcal{M}$ be a class of monomorphisms closed under isos and composition.

1. The object A has smooth $\mathcal{M}$-subobjects if $\mathrm{Sub}_{\mathcal{M}}(A)$ is a dcpo with $\bot$ where directed joins and $\bot$ are given by colimits.
2. The class $\mathcal{M}$ is smooth if every object has smooth subobjects.

Directed joins and $\bot$ given by colimits:

- initial object 0 exists, and $0 \rightarrowtail A$ lies in $\mathcal{M}$ (empty colimit)
- for $D \subseteq \mathrm{Sub}_{\mathcal{M}}(A)$ directed:



$a_{u,v}$ witnesses $u \leq v$ in D

$C = \mathrm{colim}_{u \in D} A_u$

$m = \bigvee D$

Examples of Smooth Classes

all $0 \to A$ are strong monos

1. (strong) monos in every lfp category with a simple initial object, e.g. sets, posets, graphs, monoids, vector spaces (but not rings)

e embedding if $\exists \hat{e}.\ \hat{e} \cdot e = \mathrm{id}$ and $e \cdot \hat{e} \leq \mathrm{id}$.

2. embeddings in dcpos with $\perp$ + continuous maps (but all monos are not a smooth class)
3. (strong) monos in metric spaces + non-expansive maps

represent closed subspaces

4. strong monos in complete metric spaces + non-expansive maps (but all monos are not a smooth class)

Ingredient 3: Recursive Coalgebras

Recursive Coalgebras

Definition

A coalgebra $C \xrightarrow{\gamma} FC$ is recursive if for every algebra $FA \xrightarrow{\alpha} A$ there exists a unique coalgebra-to-algebra morphism

$$\begin{array}{ccc} C & \xrightarrow{h} & A \\ {\scriptstyle\gamma}\big\downarrow & & \big\uparrow{\scriptstyle\alpha} \\ FC & \xrightarrow{Fh} & FA \end{array}$$

- Osius (1974): first studies them in connection with well-founded coalgebras for the power-set functor
- Taylor (1995, 1999, 2021): considers them for general functors as coalgebras obeying the recursion scheme;
  General Recursion Theorem: well-founded $\implies$ recursive.
- Capretta, Uustalu and Vene (2006): constructions of recursive coalgebras; semantic treatment of recursive divide-and-conquer programs

Key Properties of Recursive Coalgebras

- $0 \to F0$ is recursive
- $C \xrightarrow{\gamma} FC$ recursive $\implies$ $FC \xrightarrow{F\gamma} FFC$ recursive
- recursive coalgebras are closed under all colimits
  (in the category of F-coalgebras)
- $C \xrightarrow{\gamma} FC$ recursive and γ iso $\implies$ $FC \xrightarrow{\gamma^{-1}} C$ initial algebra
- Every coalgebra $W_i \xrightarrow{w_{i,i+1}} FW_i$ in the initial-algebra chain is recursive.

Combining the Ingredients

Initial Algebra Theorem

Theorem

Let $FA \overset{m}{\rightarrowtail} A$ be an $\mathcal{M}$-pre-fixed point of an endofunctor F preserving $\mathcal{M}$. If A has smooth $\mathcal{M}$-subobjects, then F has an initial algebra.
(which is an $\mathcal{M}$-subalgebra of $FA \overset{m}{\rightarrowtail} A$)

Proof. Use Pataraia Induction on

$$f : \mathrm{Sub}_{\mathcal{M}}(A) \to \mathrm{Sub}_{\mathcal{M}}(A), \quad f\left(B \overset{u}{\rightarrowtail} A\right) = \left(FB \overset{Fu}{\rightarrowtail} FA \overset{m}{\rightarrowtail} A\right),$$

$$S = \left\{ B \overset{u}{\rightarrowtail} A : \exists\ B \overset{\beta}{\rightarrowtail} FB \text{ recursive s.th. } \begin{array}{ccc} B & \overset{u}{\rightarrowtail} & A \\ {\scriptstyle\beta}\downarrow & & \uparrow{\scriptstyle m} \\ FB & \overset{Fu}{\rightarrowtail} & FA \end{array} \right\}.$$

all $u \in \mathrm{Sub}_{\mathcal{M}}(A)$ s.th. $u \leq f(u)$
via a recursive coalgebra

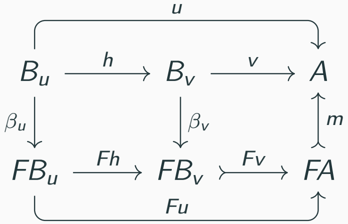
Initial Algebra Theorem (Proof continued)

$$S = \{B \overset{u}{\rightarrowtail} A : \exists\ B \overset{\beta}{\rightarrowtail} FB \text{ recursive s.th. } u = f(u) \cdot \beta.\}$$

1. $\bot \in S$: clearly $u \colon 0 \rightarrowtail A$ is in S
2. S closed under f: if $u \in S$, then $f(u) \in S$ since

   $$f(u) = m \cdot Fu = m \cdot F(f(u) \cdot \beta) = m \cdot F(f(u)) \cdot F\beta = f(f(u)) \cdot F\beta$$

   (β recursive $\implies$ $F\beta$ recursive)
3. S closed under directed joins: given $D \subseteq S$ directed, $u \leq v$ in D is necessarily witnessed by a coalgebra homomorphism $h \colon B_u \to B_v$:



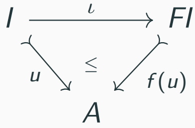
Now let $v = \bigvee D$. Then

- $B_v = \operatorname{colim}_{u \in D} B_u$
- v uniquely induced by colimit
- $B_v \overset{\beta_v}{\longrightarrow} FB_v$ recursive
- v coalgebra-to-algebra morphism

Therefore $v \in S$.

Initial Algebra Theorem (Proof continued)

- By Parataia Induction: $\mu f \in S$, ... let's denote it $I \overset{u}{\rightarrowtail} A$.
- There is a recursive coalgebra $I \xrightarrow{\iota} FI$ witnessing $u \leq f(u)$:



- Since $u = f(u)$ in $\mathrm{Sub}_{\mathcal{M}}(A)$, ι is an iso.
- Therefore $FI \xrightarrow{\iota^{-1}} I$ is an initial algebra. □

Initial Algebra Theorem

Corollary

Let $\mathcal{C}$ have a smooth class $\mathcal{M}$ of monomorphisms.
For $F\colon \mathcal{C} \to \mathcal{C}$ preserving $\mathcal{M}$ tfae:

- an initial algebra exists,
- a fixed point exists,
- an $\mathcal{M}$-pre-fixed point exists.

Moreover, if these hold, then μF is an $\mathcal{M}$-subalgebra of every $\mathcal{M}$-pre-fixed point of F.

Conclusions and Further Work

- A constructive proof of Trnková et al.'s initial algebra theorem.
  (omitting the transfinite initial-algebra chain)

In the paper:

- A streamlined (non-constructive) proof of the original theorem (with initial-algebra chain) based on Zermelo's theorem.
- An application to categories enriched in dcpos with $\perp$: coincidence of initial algebra and terminal coalgebra, if they exist.

Appendix ...

Dcpo Enriched Categories

Corollary

Let $\mathcal{C}$ be enriched in the category of dcpos with $\bot$ and continuous maps with composition strict in both arguments. For $F\colon \mathcal{C} \to \mathcal{C}$ locally monotone tfae:

- an initial algebra exists,
- a terminal coalgebra exists,
- a fixed point exists,
- a pre-fixed point carried by an embedding exist.

Moreover: $FI \xrightarrow{\iota} I$ initial algebra $\implies$ $I \xrightarrow{\iota^{-1}} FI$ terminal coalgebra.

Theorem

Let $\mathcal{C}$ be enriched in the category of dcpos with $\bot$ and continuous maps with composition left-strict. Let $F\colon \mathcal{C} \to \mathcal{C}$ be locally monotone.

Then: $FI \xrightarrow{\iota} I$ initial algebra $\implies$ $I \xrightarrow{\iota^{-1}} FI$ terminal coalgebra.

Freyd proved this for locally continuous functors using Kleene's theorem.

Here we use Pataraia Induction.