Which categories are varieties?

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Theorem 1. (Lawvere 1963) A category is equivalent to a variety iff it has

(1) finite limits and coequalizers,

(2) effective equivalence relations, and

(3) an abstractly finite, regularly projective regular generator G.

(3) describes properties of a free algebra on one generator.

Definition 1. An object G is *abstractly finite* if it has copowers, and every morphism to a copower $G \to X \cdot G$ factorizes through a finite subcopower $Y \cdot G \to X \cdot G$.

Recall that an object K in a category $\mathcal{K}$ is finitely presentable (finitely generated) if its hom-functor $\mathcal{K}(K, -) : \mathcal{K} \to \textbf{\textit{Set}}$ preserves directed colimits (directed colimits of monomorphisms).

Finitely presentable $\Rightarrow$ finitely generated $\Rightarrow$ abstractly finite.

In a variety, G is finitely presentable. But, in general, an abstractly finite algebra does not need to be finitely generated.

In sets, or in vector spaces, abstractly finite = finitely generated = finitely presentable. But a unary algebra (with one operation) is abstractly finite iff it has finitely many connected components. Hence it does not need to be finitely generated.

Example 1. No non-empty cpo is finitely generated. But finite cpo's are abstractly finite. They even have the property that every morphism to a coproduct $\coprod_{i \in I} K_i$ factorizes essentially uniquely through a finite subcoproduct $\coprod_{j \in J} K_j$.

Definition 2. An object G is a *regular generator* if it has copowers and for every object K the canonical morphism $\mathcal{K}(G, K) \cdot G \to K$ is a regular epimorphism.

Definition 3. An object G is *regularly projective* if $\mathcal{K}(G, -)$ preserves regular epimorphisms.

Definition 4. A relation on an object K is represented by a jointly monic pair $r_1, r_2 : R \to K$. It is an *equivalence relation* if $\{(r_1 f, r_2 f); f : X \to R\}$ is an equivalence relation on $\mathcal{K}(X, K)$ for every object X.
It is called *effective* if it is a kernel pair of some morphism.

Lemma 1. In a category with kernel pairs, an object is regularly projective iff its hom-functor preserves coequalizers of effective equivalence relations.

Definition 5. An object is called *effective* if its hom-functor preserves coequalizers of equivalence relations.

Theorem 2. A category is equivalent to a variety iff it has reflexive coequalizers and an abstractly finite, effective regular generator.

Our aim is to generalize this characterization to many-sorted varieties which are important in software specifications.

Definition 6. A set $\mathcal{G}$ of objects is *abstractly finite* if all coproducts of $\mathcal{G}$-objects exist, and every morphism $G \to \coprod_{i\in I} G_i$ with G and all G_i in $\mathcal{G}$ factorizes through a finite subcoproduct $\coprod_{j\in J} G_j$.

Definition 7. A set $\mathcal{G}$ of objects is a *regular generator* if all coproducts of $\mathcal{G}$-objects exist and for every object K the canonical morphism $\coprod_{G\in\mathcal{G}} \coprod_{f:G\to K} G \to K$ is a regular epimorphism.

Theorem 3. A category is equivalent to a finitely-sorted variety iff it has reflexive coequalizers and an abstractly finite, regular generator consisting of finitely many effective objects.

The proof is based on the fact that, if S is finite, then every finitely bounded $\boldsymbol{Set}^S \to \boldsymbol{Set}^S$ is finitary (Adámek, Milius, Sousa, Wissmann, 2019). This means that every $x \in FX$ belongs to FY for a finite subobject Y of X.

This characterization cannot be extended to infinitely-sorted varieties.

Example 2. Let $\mathcal{K}$ be the full subcategory of $\boldsymbol{Set}^{\mathbb{N}}$ consisting of the terminal object $1 = (1, 1, 1 \dots)$ and all objects $(X_n)_{n \in \mathbb{N}}$ such that for some $k \in \mathbb{N}$ we have $X_n \neq \emptyset$ iff $n < k$. $\mathcal{K}$ is closed under coequalizers in $\boldsymbol{Set}^{\mathbb{N}}$. But not under colimits of chains: consider the chain of inclusions of $X^k = (X_n^k)$, $k < \omega$ where $X_n^k = \{0, 1\}$ for $n \leq k$, else $\emptyset$. Then $\operatorname{colim}_{k<\omega} X^k = 1$ in $\mathcal{K}$. And the only object of $\mathcal{K}$ that preserves this colimit is $(\emptyset, \emptyset, \emptyset \dots)$. Hence $\mathcal{K}$ cannot be equivalent to an $\mathbb{N}$-sorted variety.
However, $\mathcal{K}$ has the abstractly finite regular generator $\{G^k\}$ $k \in \mathbb{N}$ where $G_n^k = 1$ for $n \leq k$, else $\emptyset$. Every morphism $f : G^k \to \coprod_{i \in I} G^{k_i}$ has the property that $k \leq k_i$ for some i, thus f factorizes through the coproduct injection of G^{k_i}. (This factorization is not essentially unique.) The verification that each G^k is effective and that they form a regular generator is easy.

Definition 8. An object is called *perfectly presentable* if its hom-functor preserves directed colimits and reflexive coequalizers.

Since equivalence relations are reflexive, perfectly presentable $\Rightarrow$ effective and finitely presentable.

Definition 9. A set $\mathcal{G}$ of objects is a *strong generator* if all coproducts of $\mathcal{G}$-objects exist and every object K is an extremal quotient of a coproduct of $\mathcal{G}$-objects.

Theorem 4. A category is equivalent to a many-sorted variety iff it is cocomplete and has a strong generator consisting of perfectly presentable objects.

If $\mathcal{K}$ has kernel pairs then a strong generator consisting of regularly projective objects is a regular generator.

The category of cpo's is complete, cocomplete and has an abstractly finite, strong generator consisting of finitely many objects. But it is not locally finitely presentable.