On Doctrines and Cartesian Bicategories

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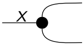
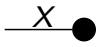
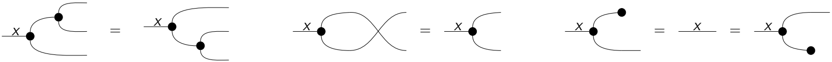
Fox's theorem for cartesian categories[1]

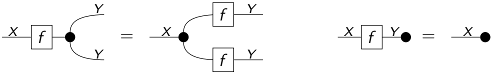
A symmetric monoidal category $(\mathbb{C}, \otimes, I)$ is cartesian if and only if every object X is equipped with morphisms



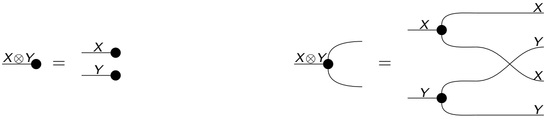
$: X \to X \otimes X$ and 

$: X \to I$ such that

1. 

2. For all $f : X \to Y$: 

3. The choice of comonoid on every object is coherent with the monoidal structure in the sense that



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[1] Fox, "Coalgebras and cartesian categories", 1976.

Lawvere theories for terms

Let $\mathcal{L} = (\Sigma, \mathbb{P})$ and $\mathcal{T}$ be a theory in regular logic with equality.
The *Lawvere Theory* generated by Σ is a category L_Σ.

- Objects: natural numbers.
- Morphisms $n \to m$: tuples $\langle t_1, \ldots, t_m \rangle$ where $Var(t_i) \subseteq \{x_1, \ldots, x_n\}$.
- Composition: for $n \xrightarrow{\langle t_1, \ldots, t_m \rangle} m \xrightarrow{\langle s_1, \ldots, s_l \rangle} l$

$$\langle s_1, \ldots, s_l \rangle \circ \langle t_1, \ldots t_m \rangle = \langle s_1[\vec{t_i}/\vec{x_i}], \ldots, s_l[\vec{t_i}/\vec{x_i}] \rangle$$

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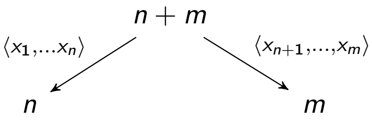
L_Σ is cartesian: $n \times m = n + m$ with projections



The Lindenbaum-Tarski doctrine

For $n \in \mathbb{N}$ define

$$LT(n) = \{[\phi] \mid \phi \text{ formula in } \mathcal{L} \text{ with free variables in } \{x_1, \dots, x_n\}\ \}$$

where $[\phi] = [\phi']$ if and only if $\phi \dashv\vdash \phi'$ in the theory $\mathcal{T}$.
Set $[\phi] \leq [\psi]$ if and only if $\phi \vdash \psi$ in $\mathcal{T}$.

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$$\mathsf{L}_\Sigma^{\mathsf{op}} \xrightarrow{LT} \mathsf{InfSL}$$

$$\begin{array}{ccc} m & \longmapsto & LT(m) \\ {\scriptstyle \langle t_1,\ldots,t_m \rangle} \uparrow & & \downarrow {\scriptstyle \cdot[\vec{t_i}/\vec{x_i}]} \\ n & \longmapsto & LT(n) \end{array}$$

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\end{array}$$

Notice: for $\pi_1 = \langle x_1, \dots, x_n\rangle \colon n + m \to n$ in L_{Σ},
$LT(\pi_1) \colon LT(n) \to LT(n+m)$ has a left adjoint $\exists_{\pi_1} \colon LT(n+m) \to LT(n)$

$$\exists_{\pi_1}(\phi) = \exists x_m \dots \exists x_{n+1}. \phi$$

Elementary existential doctrines

An elementary existential doctrine[2] is a functor $P\colon \mathbb{C}^{\mathrm{op}} \to \mathsf{InfSL}$, with $\mathbb{C}$ cartesian, such that

- for all $A \in \mathbb{C}$ there is an element $\delta_A \in P(A \times A)$ satisfying certain adjoint conditions,
- for all $\pi\colon X \times A \to A$ projection, P_π has a left adjoint $\exists_\pi\colon P(X \times A) \to P(A)$ satisfying certain conditions.

[2] Maietti and Rosolini, "Quotient Completion for the Foundation of Constructive Mathematics", 2013.

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Example

Powerset $\mathcal{P}\colon \mathsf{Set}^{\mathrm{op}} \to \mathsf{InfSL}$. For $f\colon X \to Y$, $Z \in \mathcal{P}(Y)$:

$$\mathcal{P}(f)(Z) = \{x \in X \mid f(x) \in Z\} \in \mathcal{P}(X).$$

- $\delta_A = \{(a, a) \mid a \in A\} \in \mathcal{P}(A \times A)$
- For $\pi\colon X \times A \to A$ projection:

$$\begin{array}{ccc} \mathcal{P}(X \times A) & \xrightarrow{\exists_\pi} & \mathcal{P}(A) \\ S & \longmapsto & \{a \in A \mid \exists x \in X.\, (x, a) \in S\} \end{array}$$

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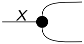
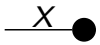
Cartesian bicategories

A cartesian bicategory[3] is a Poset-enriched, symmetric monoidal category $(\mathbb{B}, \otimes, I)$ where every object $X \in \mathbb{B}$ is equipped with morphisms



$: X \to X \otimes X$ and 

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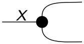
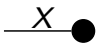
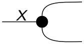
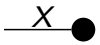
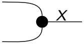
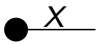
1. 

and 

form a cocommutative comonoid
2. 

and 

have right adjoints 

and 

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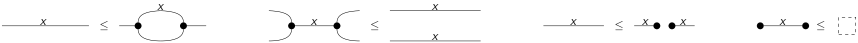
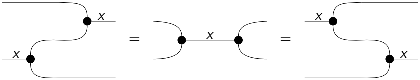
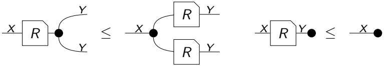


3. The Frobenius law holds: 

4. For $R \colon X \to Y$: 

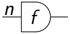
5. The choice of comonoid is coherent with the monoidal structure.

[3] Carboni and Walters, "Cartesian Bicategories I", 1987.

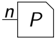
The cartesian bicategory generated by a regular theory

$\mathsf{CB}_{\Sigma,\mathbb{P}}$: the free cartesian bicategory whose objects are natural numbers and generators for morphisms are given by the following rules:

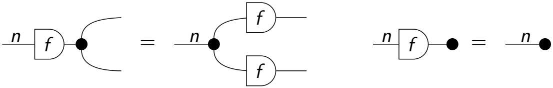
$$\frac{f \in \Sigma \quad ar(f) = n}{\qquad : n \to 1}\,\Sigma$$



$$\frac{P \in \mathbb{P} \quad ar(P) = n}{\qquad : n \to 0}\,\mathbb{P}$$

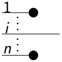


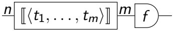
where we require that:



Terms and formulae in $\mathrm{CB}_{\Sigma,\mathbb{P}}$

Interpretation of terms and formulae, where $Var(t_j) \subseteq \{x_1, \ldots, x_n\}$:

$[\![x_i]\!] =$ 

$[\![f\langle t_1, \ldots, t_m\rangle]\!] =$ 

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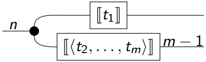
$[\![\langle t_1, \ldots, t_m\rangle]\!] =$ 

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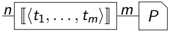
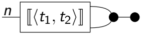
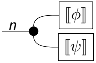
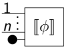
$[\![P\langle t_1, \ldots, t_m\rangle]\!] =$ 

$[\![t_1 = t_2]\!] =$ 

$[\![\phi \wedge \psi]\!] =$ 

$FreeVar(\phi) = FreeVar(\psi) \subseteq \{x_1, \ldots, x_n\}$

$[\![\exists x_{n+1}.\, \phi]\!] =$ 

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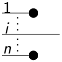
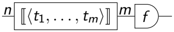
$[\![x_i]\!] =$ 

$[\![f\langle t_1, \ldots, t_m\rangle]\!] =$ 

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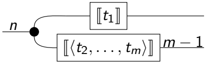
$[\![\langle t_1, \ldots, t_m\rangle]\!] =$ 

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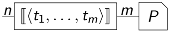
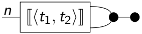
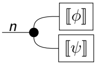
$[\![P\langle t_1, \ldots, t_m\rangle]\!] =$ 

$[\![t_1 = t_2]\!] =$ 

$[\![\phi \wedge \psi]\!] =$ 

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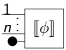
$[\![\exists x_{n+1}.\, \phi]\!] =$ 

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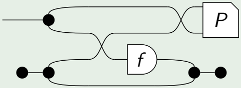
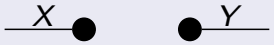
Example

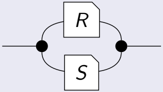
$[\![\exists x_2.\, (P(x_2, x_1) \wedge f(x_1) = x_2)]\!] =$ 

From cartesian bicategories to doctrines

Lemma

Let $\mathbb{B}$ be a cartesian bicategory and $X, Y \in \mathbb{B}$. The poset $\text{Hom}_{\mathbb{B}}(X, Y)$ *has a top element given by* 

and the meet of $R, S \colon X \to Y$ is:



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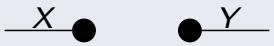
Let $\mathbb{B}$ be a cartesian bicategory and $X, Y \in \mathbb{B}$. The poset $\mathrm{Hom}_{\mathbb{B}}(X, Y)$ has a top element given by 

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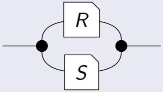

$$\mathbb{B}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathbb{B}}(-, I)} \mathsf{Set}$$

$$\begin{array}{ccc} Y & \longmapsto & \mathrm{Hom}_{\mathbb{B}}(Y, I) \\ R \uparrow & & \downarrow - \circ R \\ X & \longmapsto & \mathrm{Hom}_{\mathbb{B}}(X, I) \end{array}$$

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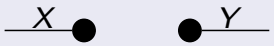
Let $\mathbb{B}$ be a cartesian bicategory and $X, Y \in \mathbb{B}$. The poset $\mathrm{Hom}_{\mathbb{B}}(X, Y)$ has a top element given by 

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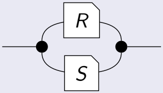

$$(\mathrm{Map}\,\mathbb{B})^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathbb{B}}(-,I)} \mathrm{InfSL}$$

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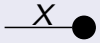
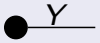
Let $\mathbb{B}$ *be a cartesian bicategory and* $X, Y \in \mathbb{B}$*. The poset* $\mathrm{Hom}_{\mathbb{B}}(X, Y)$ *has a top element given by* 



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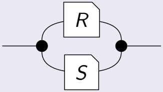

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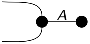
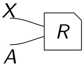
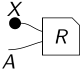
$\mathcal{R}(\mathbb{B}) = \mathrm{Hom}_{\mathbb{B}}(-, I)\colon (\mathrm{Map}\,\mathbb{B})^{\mathrm{op}} \to \mathrm{InfSL}$ is an elementary existential doctrine where, for $\pi\colon X \otimes A \to A$ projection:

$\delta_A^{\mathcal{R}(\mathbb{B})} =$ 

$\in \mathrm{Hom}_{\mathbb{B}}(A \otimes A, I)$ $\quad \exists_\pi($ 

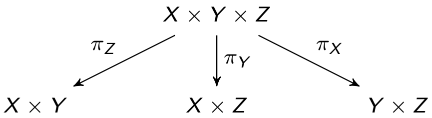
$) =$ 

From doctrines to cartesian bicategories[4]

If $P \colon \mathbb{C}^{\mathrm{op}} \to \mathsf{InfSL}$ is an EED, then the category $\mathscr{A}_P$ is a CBC, where:

- Objects: those of $\mathbb{C}$.
- Morphisms $X \to Y$: elements of $P(X \times Y)$.
- Composition of $f \in \mathrm{Hom}_{\mathscr{A}_P}(X, Y) = P(X \times Y)$ and $g \in \mathrm{Hom}_{\mathscr{A}_P}(Y, Z) = P(Y \times Z)$:

$$g \circ f = \exists_{\pi_Y}(P_{\pi_Z}(f) \wedge P_{\pi_X}(g))$$



[4] Maietti and Rosolini, "Unifying Exact Completions", 2015.

An adjunction

$$\mathsf{EED} \underset{\mathcal{R}}{\overset{\mathcal{L}}{\underset{\bot}{\rightleftarrows}}} \mathsf{CBC}$$

$$\begin{cases} \mathcal{L}(P) = \mathscr{A}_P \\ \mathcal{R}(\mathbb{B}) = \mathsf{Hom}_{\mathbb{B}}(-, I) \colon (\mathsf{Map}\,\mathbb{B})^{\mathsf{op}} \to \mathsf{InfSL} \end{cases}$$

An adjunction

$$\mathsf{EED} \underset{\mathcal{R}}{\overset{\mathcal{L}}{\rightleftarrows}} \mathsf{CBC} \qquad (\mathcal{L} \dashv \mathcal{R}) \qquad \begin{cases} \mathcal{L}(P) = \mathscr{A}_P \\ \mathcal{R}(\mathbb{B}) = \mathrm{Hom}_{\mathbb{B}}(-, I)\colon (\mathrm{Map}\,\mathbb{B})^{\mathrm{op}} \to \mathsf{InfSL} \end{cases}$$

$\mathcal{L}$ is not faithful. Consider $\Sigma_1 = \{f\}$, $\Sigma_2 = \{g_1, g_2\}$ and the two doctrines:

$$\mathsf{L}_{\Sigma_1}^{\mathrm{op}} \xrightarrow{LT} \mathsf{InfSL} \quad \text{and} \quad \mathsf{L}_{\Sigma_2}^{\mathrm{op}} \xrightarrow{Q^{\mathrm{op}}} \mathsf{L}_{\Sigma_1}^{\mathrm{op}} \xrightarrow{LT} \mathsf{InfSL}$$

where $Q(n) = n$ and $Q(g_1) = f = Q(g_2)$.

An adjunction

$$\mathrm{EED} \underset{\mathcal{R}}{\overset{\mathcal{L}}{\rightleftarrows}} \mathrm{CBC} \qquad (\mathcal{L} \dashv \mathcal{R})$$

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where $Q(n) = n$ and $Q(g_1) = f = Q(g_2)$. Then:

$$\begin{array}{ccc} \mathrm{L}_{\Sigma_2}^{\mathrm{op}} & \xrightarrow{LT \circ Q^{\mathrm{op}}} & \mathrm{InfSL} \\ m & \longmapsto & LT(m) \\ {\scriptstyle \langle t_1, \ldots, t_m \rangle} \uparrow & & \downarrow {\scriptstyle \cdot [Q(\vec{t_i})/\vec{x_i}]} \\ n & \longmapsto & LT(n) \end{array}$$

$\mathcal{L}(LT) = \mathcal{L}(LT \circ Q^{\mathrm{op}})$ but they are not isomorphic as doctrines.

A Fox theorem for regular logic

To have an equivalence, we need the unit $\eta_P \colon P \to \mathcal{RL}(P)$ to be a natural isomorphism.

$$\mathcal{RL}(P) = \mathrm{Hom}_{\mathscr{A}_P}(-, I) = P(- \times I) \colon \ \mathsf{Map}(\mathscr{A}_P)^{\mathsf{op}} \to \mathsf{InfSL}\,.$$

Hence we need that $\mathbb{C} \cong \mathsf{Map}\,(\mathscr{A}_P)$. This happens if and only if:

1. P has *comprehensive diagonals*,
2. P satisfies the axiom of unique choice.[5]

[5] Maietti, Pasquali, and Rosolini, "Triposes, exact completions, and Hilbert's ε-operator", 2017.

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Proposition

Let $\mathbb{B}$ be a cartesian bicategory. Then $\mathcal{R}(\mathbb{B})$ satisfies (1) and (2).

The adjunction $\mathsf{EED} \underset{\mathcal{R}}{\overset{\mathcal{L}}{\underset{\perp}{\rightleftarrows}}} \mathsf{CBC}$ restricts to an equivalence when EED is replaced with its subcategory $\overline{\mathsf{EED}}$ of doctrines satisfying (1), (2).

[5] Maietti, Pasquali, and Rosolini, "Triposes, exact completions, and Hilbert's ε-operator", 2017.