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Closure Hyperdoctrines

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Recent interest in modal logic modeling the notion of "proximity", such as the *Spatial Logic for Closure Spaces* (SLCS) introduced by Ciancia et al. [2, 1].

The central concept is that of *closure space* or *pretopological space*.

Definition ([1, 2, 4])

A *closure space* is a pair $(X, \mathfrak{c})$ where X is a set and $\mathfrak{c}$ is a function $\mathcal{P}(X) \to \mathcal{P}(X)$ such that, for any A and $B \subset X$:

- $\mathfrak{c}(\emptyset) = \emptyset$;
- $A \subset \mathfrak{c}(A)$;
- $\mathfrak{c}(A \cup B) = \mathfrak{c}(A) \cup \mathfrak{c}(B)$.

The spatial "until" operator

In a closure space we can define the *until operator* $\mathfrak{U}$:

Definition

Give a closure space $(X, \mathfrak{c})$ and two subset A and B, we define the set $A\mathfrak{U}B$ as

$$\{x \in A \mid \exists C \subset A.(x \in C \wedge ((\mathfrak{c}(C) \cap (X \smallsetminus A)) \subset B))\}$$

Intuitively, if $\mathfrak{c}(A)$ is the set of points "reachable" from A, then $A\mathfrak{U}B$ is the subset of A from which there is no way out without passing through B.

Aims of this work

The main aim of this work is providing a theoretical framework for investigating the logical aspects of (pre)closure spaces. Namely, we

1. introduce the new notion of *closure (hyper)doctrine*
2. show that this notion covers many others situations besides pretopological spaces;
3. provide a syntax and a sequent calculus for a logic endowed with a notion of nearness through a closure operator;
4. provide a categorical semantics for this logic, by means of closure (hyper)doctrines;
5. prove a completeness theorem for such a semantics.

Definition

Let $\mathbf{C}$ be a category with finite products. An *elementary hyperdoctrine* on $\mathbf{C}$ is a functor $\mathscr{P} : \mathbf{C}^{op} \to \mathbf{HA}$ (the category of Heyting algebras) such that for each arrow $f : C \to D$, $\mathscr{P}_f : \mathscr{P}(D) \to \mathscr{P}(C)$ has a left and right adjoint $\exists_f$ and $\forall_f$ satysfying

$$\exists_{\pi_{C'}} \circ \mathscr{P}_{1_D \times f} = \mathscr{P}_f \circ \exists_{\pi_C} \quad \forall_{\pi_{C'}} \circ \mathscr{P}_{1_D \times f} = \mathscr{P}_f \circ \forall_{\pi_C}$$

Given two elementary hyperdoctrines $\mathscr{P} : \mathbf{C}^{op} \to \mathbf{HA}$ and $\mathscr{S} : \mathbf{D}^{op} \to \mathbf{HA}$, a morphism $\mathscr{P} \to \mathscr{S}$ is a couple $(\mathscr{F}, \eta)$ where $\mathscr{F} : \mathbf{C} \to \mathbf{D}$ is a product preserving functor and η is a natural transformation $\mathscr{P} \to \mathscr{S} \circ \mathscr{F}^{op}$ preserving $\exists_{\Delta_C}(\top)$ (the *fibered equality* at C) and quantifiers.

Ementary hyperdoctrines provide semantics for (multi-sorted) full FOL with equality.
We can weaken it in various way:

- **doctrine:** functor valued in Heyting or boolean algebras or meet semilattices, suited for propositional logic (base category may not have cartesian products);
- **existential doctrine:** functor valued in meet semilattices or in bounded lattices, with the existential quantifier satisfying *Frobenius reciprocity*:

$$\exists_f(\mathscr{P}_f(\beta) \wedge \alpha) = \beta \wedge \exists_f(\alpha)$$

Closure operators on an (hyper)doctrine

Definition

A *closure operator* on a hyperdoctrine $\mathscr{P}$ is a family of monotone functions $\mathfrak{c}_C : \mathscr{P}(C) \to \mathscr{P}(C)$ indexed by the objects of $\mathbf{C}$ s.t.:

- $1_{\mathcal{P}(C)} \leq \mathfrak{c}_C$;
- $\mathfrak{c}_C \circ \mathscr{P}_f \leq \mathscr{P}_f \circ \mathfrak{c}_D$ for any arrow $f : C \to D$.

A *closure hyperdoctrine* is a couple $(\mathscr{P}, \mathfrak{c})$ formed by an hyperdoctrine and a closure operator on it.

We can mimic this definition for other kinds of doctrines getting *closure doctrines*, *closure existential doctrines*, etc…
We can ask other properties for $\mathfrak{c}$, like (as in the case of SLCS) *additivity* and *groundedness*:

$$\mathfrak{c}_C(\alpha \vee \beta) = \mathfrak{c}_C(\alpha) \vee \mathfrak{c}_C(\beta) \qquad \mathfrak{c}_C(\bot) = \bot$$

Definition

A morphism $(\mathscr{P}, \mathfrak{c}) \to (\mathscr{S}, \mathfrak{d})$ between two closure hyperdoctrines $\mathscr{P} : \mathbf{C}^{op} \to \mathbf{HA}$ e $\mathscr{S} : \mathbf{D}^{op} \to \mathbf{HA}$ is an arrow of hyperdoctrines $(\mathscr{F}, \eta)$ between $\mathscr{P}$ and $\mathscr{S}$ such that

$$\mathfrak{d}_{\mathscr{F}(C)} \circ \eta_C \leq \eta_C \circ \mathfrak{c}_C$$

$(\mathscr{F}, \eta)$ is *open* if equality holds.
We will denote by **cEHD** the category of closure hyperdoctrines.

We can define similar categories of *closure doctrines*, *closure existential doctrines*, etc. . .

SLCS

We can use the usual power set functor in order to define a closure hyperdoctrine on pretopological spaces.
Let $\mathscr{P}(X, c) := 2^X$ and set

$$\mathfrak{c}_{(X,c)} : 2^X \to 2^X$$
$$A \mapsto c(A)$$

The semantics in this closure hyperdoctrine gives us back the SLCS's semantics developed in [1, 2].

Example: Fuzzy sets

Fuzzy sets

The category of *fuzzy set* has as objects, couples (A, α) where A is a set and $\alpha \to [0, 1]$ a function. An arrow $(A, \alpha) \to (B, \beta)$ is a function such that $\alpha(x) \leq \beta(f(x))$. A *fuzzy subset of* (A, α) is a function $\xi : A \to [0, 1]$ with the property that $\xi(x) \leq \alpha(x)$.
Assigning to (A, α) the set of its fuzzy subsets gives an elementary hyperdoctrine.
Let now $\mathcal{E}$ be a family of weights $\epsilon_{(A,\alpha)} : (A, \alpha) \to [0, 1]$, we can define

$$\mathfrak{c}_{(A,\alpha)}(\xi)(x) := \inf\{\xi(x) + \epsilon(x), \alpha(x)\}$$

In this way we get a closure operator that is additive but doesn't preserve the bottom subset.

Example: coalgebras

Discrete probability space

For a set X let $\mathscr{D}(X)$ be the set of probability measures on 2^X, a *coalgebra* for $\mathscr{D}$ is a function $\gamma_X : X \to \mathscr{D}(X)$.
Let $\mathscr{P}((X, \gamma_X)) := 2^X$ and fix a $p \in [0,1]$, the family given by:

$$\mathfrak{c}_{X,p} : 2^X \to 2^X \qquad A \mapsto A \cup \{x \in X \mid p \leq \gamma_X(x)(A)\}$$

is a closure operator.

Remark

Using the notion of predicate liftings (see Jacobs and Sokolova [6]), this example can be seen an instance of a general schema for many categories of coalgebras.
In general, categories of coalgebras do not have products, so we get only a doctrine.

Definition

Let Σ be a first order signature, a *context* Γ is a finite list $[x_i : \sigma_i]_{i=1}^n$ of typed variables. The rules for contexts and well-formed formulae for a signature Σ are the usual ones ([5]) plus:

$$\frac{\Gamma \vdash \phi : \mathsf{Prop}}{\Gamma \vdash \mathcal{C}(\phi) : \mathsf{Prop}}\ \mathcal{C}\text{-F} \qquad \frac{\Gamma \vdash \phi : \mathsf{Prop} \qquad \Gamma \vdash \psi : \mathsf{Prop}}{\Gamma \vdash \phi \mathcal{U} \psi : \mathsf{Prop}}\ \mathcal{U}\text{-F}$$

- ϕ such that $\Gamma \vdash \phi : \mathsf{Prop}$ means the "region" of Γ composed by points satisfying ϕ;
- $\mathcal{C}(\phi)$ is means the set of points "near" ϕ;
- $\phi \mathcal{U} \psi$ (to be read "ϕ until ψ") means the subregion of ϕ from which there is no "escape" without passing through ψ.

A logic for proximity: Sequent calculus

We add to the usual rules of (intuitionistic) sequent calculus the following rules for $\mathcal{C}$:

$$\frac{}{\Gamma \mid \Phi, \phi \vdash \mathcal{C}(\phi)}\ \textsc{Cl-1} \qquad \frac{\Gamma \mid \Phi, \phi \vdash \psi}{\Gamma \mid \Phi, \mathcal{C}(\phi) \vdash \mathcal{C}(\psi)}\ \textsc{Cl-2}$$

and for $\mathcal{U}$:

$$\frac{\Gamma \mid \Phi, \varphi \vdash \phi \qquad \Gamma \mid \Phi, \mathcal{C}(\varphi), \neg\phi \vdash \psi}{\Gamma \mid \Phi, \varphi \vdash \phi \mathcal{U} \psi}\ \mathcal{U}\text{-I}$$

$$\frac{\text{for all } \varphi \in \mathsf{u}_{(\Gamma,\Phi)}(\phi,\psi) : \Gamma \mid \Phi, \varphi \vdash \theta}{\Gamma \mid \Phi, \phi \mathcal{U} \psi \vdash \theta}\ \mathcal{U}\text{-E}$$

where:

$$\mathsf{u}_{(\Gamma,\Phi)}(\phi,\psi) := \{\varphi \text{ such that } \Gamma \mid \Phi, \varphi \vdash \phi, \Gamma \mid \Phi, \mathcal{C}(\varphi), \neg\varphi \vdash \psi\}$$

Remark

In order to get a logic more similar to SLCS [2, 1] we can add the rules:

$$\frac{}{\Gamma \mid \Phi, \mathcal{C}(\bot) \vdash \bot} \textsc{Cl-3}$$

$$\frac{}{\Gamma \mid \Phi, \mathcal{C}(\phi \vee \psi) \vdash \mathcal{C}(\phi) \vee \mathcal{C}(\psi)} \textsc{Cl-4}$$

$$\frac{}{\Gamma \mid \Phi, \mathcal{C}(\phi) \vee \mathcal{C}(\psi) \vdash \mathcal{C}(\phi \vee \psi)} \textsc{Cl-5}$$

Adding these rules will be reflected by additional algebraic properties of the closure operator we will use to interpret $\mathcal{C}$.

We will now introduce a *syntactic hyperdoctrine* in order to define models.

Definition

Given a signature Σ, its *classifying category* is the category $\mathbf{Cl}(\Sigma)$ in which:

- objects are contexts;
- Given $\Gamma := [x_i : \sigma_i]_{i=1}^n$, $\Delta = [y_i : \tau_i]_{i=1}^m$ an arrow $\Gamma \to \Delta$ is a m-uple of terms $(T_1, ..., T_m)$ such that $\Gamma \vdash T_i : \tau_i$ for any i;
- composition is given by substitution.

The Syntactic Hyperdoctrine

Definition

For any context Γ we define $\mathbf{Form}_\Sigma(\Gamma)$ to be the set of formulae ϕ such that $\Gamma \vdash \phi : \mathsf{Prop}$. ϕ and $\psi \in \mathbf{Form}_\Sigma(\Gamma)$ are *provably equivalent* if $\Gamma \mid \psi \vdash \phi$ and $\Gamma \mid \phi \vdash \psi$, we will denote the quotient of $\mathbf{Form}_\Sigma(\Gamma)$ by this relation with $\mathcal{L}(\Sigma)(\Gamma)$, $[\phi]$ will denote the class of ϕ in it.

Remark

$\mathcal{L}(\Sigma)(\Gamma)$ equipped with the order $[\phi] \leq [\psi]$ if and only if $\Gamma \mid \phi \vdash \psi$ is derivable is an Heyting algebra.

Theorem

For any signature Σ, the functor sending Γ to $\mathcal{L}(\Sigma)(\Gamma)$ gives us an hyperdoctrine $\mathcal{L}(\Sigma)$ and $[\phi] \mapsto [\mathcal{C}(\phi)]$ is a closure operator.

Definition

A *model* in a closure hyperdoctrine $(\mathscr{P}, \mathfrak{c})$ is an open morphism $(\mathscr{M}, \mu) : (\mathcal{L}(\Sigma), \mathcal{C}) \to (\mathscr{P}, \mathfrak{c})$.
A sequent $\Gamma \mid \Phi \vdash \psi$ is *satisfied by* $(\mathscr{M}, \mu)$ if

$$\bigwedge_{\phi \in \Phi} \mu_\Gamma(\phi) \leq \mu_\Gamma(\psi)$$

Remark

Notice that there are no conditions on the image of $\phi \mathcal{U} \psi$.

Theorem

A sequent $\Gamma \mid \Phi \vdash \psi$ *is satisfied by the generic model* $(1_{\mathbf{Cl}(\Sigma)}, 1_{\mathcal{L}(\Sigma)})$ *if and only if it is derivable.*

Some remarks on $\mathcal{U}$

We have not put any condition on the interpretation of $\phi\mathcal{U}\psi$. One could wonder what kind of additional structure should be required to interpret it.

- For a model $(\mathscr{M}, \mu)$ we can ask that $\mu_\Gamma([\phi\mathcal{U}\psi])$ to be the supremum of $\mu_\Gamma(\mathsf{u}_{(\Gamma,\Phi)}(\phi,\psi))$ for any Γ.
- Or we can ask for (limited) second order quantification restricting to model in *triposes* ([7]) and define $\phi\mathcal{U}\psi$ to be a shorthand for

$$\exists\alpha \in \mathscr{P}(C)(x \in \alpha \wedge \alpha \leq \phi \wedge ((\mathcal{C}(\alpha) \wedge \neg\alpha) \leq \psi))$$

It turns out that in the case of pretopological spaces these two approaches are equivalent, but this is not true in general.

Further work

1. Provide interpretations of $\mathcal{U}$ that limit the infinitary nature of rule $\mathcal{U}$-E, maybe using some kind of fixed point operator.
2. In [1] SLCS is improved with a notion of *path* (of some shape I) and a *surrounded* operator $\mathcal{S}$ such that $\phi \mathcal{S} \psi$ models the notion of "there is no path out of ϕ that doesn't pass through ψ". We want to add this additional operator to our categorical framework.
3. Investigate connection with closure operators studied in the context of categorical topology (see, e.g. [3]).

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