How to write a coequation

Fredrik Dahlqvist and Todd Schmidt

University College London

Salzburg via London, 3 September 2021

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- ... but coequations haven't really been adopted as a practical formalism by computer scientists
- Why?

Central premise

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- Difficult for the end-user to understand what a coequation is
- Which formalism should be used in practice?

Root cause

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- Equations are given by pairs of terms
  - Terms are *finite* trees
  - Using brackets there is an unambiguous finite string representation
- Coequations typically deal with with generalised trees
  - Infinitely branching, infinite depth
  - No finite string representation
- Impossible to get a simple syntax working well in every case

Outline of the paper

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- History of the notion of coequation

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- From this extract 4 kinds of syntax
  - Coequation-as-corelation
  - Coequation-as-predicate
  - Coequation-as-equation
  - Coequation-as-modal-formula

Coequation-as-corelation

Equations

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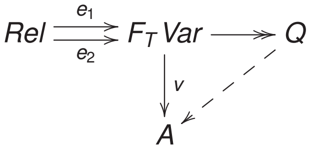
Equations are *relations* under which one can take a *quotient*



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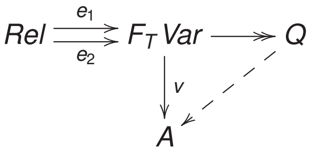
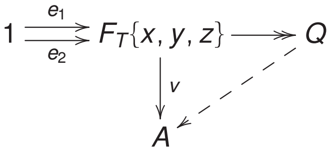


Example: semigroups

$TX = X \times X$, $Var = \{x, y, z\}$, $Rel = 1$, $e_1(*) = (xy)z$, $e_2(*) = x(yz)$



Coequations

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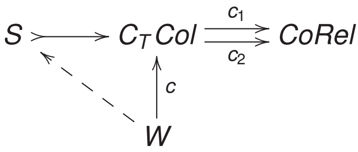
Dually, coequations are *corelations* defining a subobject



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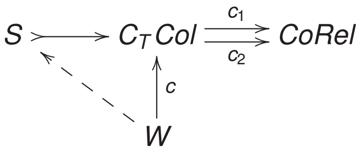
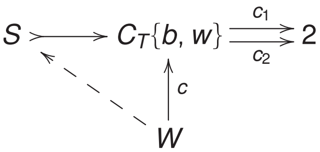


Example: deterministic binary trees $TX = X \times X$, $Col = \{b, w\}$, $CoRel = 2$, $c_1(t) = 1$ if $\text{Left}(t) = b$, $c_2(t) = 1$ if $\text{Right}(t) = b$



Coequation-as-predicate

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Two flavours: for a covarietor T, a coequation-as-predicate can be

- A subcoalgebra $Coeq \rightarrowtail C_T Col$
- A subset $Coeq \rightarrowtail U_T C_T Col$

No particular syntax, any way of describing a subcoalgebra/subset will do.

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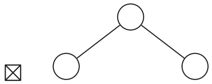
Special syntax for pattern avoidance (Gumm, Adamek and friends): $\boxtimes t$

Examples

1. For $TX = X \times X + 1$



defines the covariety of binary trees which do *not* have two halting successors.

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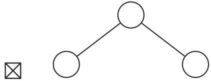

defines the covariety of binary trees which do *not* have two halting successors.

2. A T-coalgebra (V, γ) is *locally finite* if for every $v \in V$ there exists a finite subcoalgebra S of (V, γ) such that $v \in S$. The class of locally finite T-coalgebra is a covariety. By a theorems from Rutten and Adamek there must exist a coequation in ω-colours describing it.

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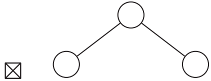

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3. The *filter functor* is not a covarietor. A generalized notion of coequation must be used. The class of topological spaces and open maps is a covariety in the class of coalgebras for the filter functor. Kurz and Rosicky present this covariety by a generalized coequation.

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- Write specifications in the usual equational format

$$\mathrm{bal}(x) + n = \mathrm{bal}(\mathrm{credit}(n, x))$$

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- Format of destructor signatures guarantee that currying is possible
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- Classify behaviours according to what the functions $\lambda n.[\![\mathrm{bal}(x) + n]\!]$ and $\lambda n.[\![\mathrm{bal}(\mathrm{credit}(n, x))]\!]$ do, then *select* those for which the classifications match up

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- Coalgebraic Goldblatt-Thomason theorem

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    - Not sure: Reason directly in terms of covariety?

Thank you.