Scientific Modelling with Coalgebra–Algebra Homomorphisms

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1 Introduction

We investigate classes of coalgebra–algebra homomorphisms of a functor F, henceforth *F-ca-homomorphisms*, that is morphisms h, without loss of generality in the category **Set**, which make the following diagram commute, where F is an endofunctor and (X, f) and (Y, g) are an F-coalgebra and F-algebra, respectively.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & FX \arrow[d, "Fh"] \\ Y & FY \arrow[l, "g"] \end{tikzcd}$$

CA-homomorphisms exists uniquely for some distinguished algebras, called *corecursive* [3], and dually for some distinguished coalgebras, called *recursive* [2]. We investigate a dual pair of classes of weaker situations, where families of canonical ca-homomorphisms exist, indexed by the solutions of a (co)recursion scheme arising from (co)monadic structure of the functor. We also demonstrate the use of these situations in the domain of scientific modelling, not so much in the *construction* of models, but in the abstract *reasoning* about them. We show that a number of such reasoning modes, both standard and advanced, nicely fit the pattern, and can thus be compared and dualized in the present framework.

2 (Co)Recursion Schemes

▶ **Definition 2.1** (Kleisli Coinduction). Let $\mathbb{T} = (T, \eta, \mu)$ be a monad. Let (X, e) be a T-coalgebra. A morphism $e^\dagger : X \to TY$ is called a *Kleisli-coinductive solution* of *equation* e, if and only if $e^\dagger = e^\dagger \circ^{\mathbf{Kl}(\mathbb{T})} e$.

▶ **Theorem 2.2** (Universality). *Given* any *T-coalgebra (X, e) and Kleisli-coinductive solution $e^\dagger : X \to TY$, there is a canonical family of extensions to T-ca-homomorphisms $(f \circ e^\dagger)$ into **all** $\mathbb{T}$-algebras (Y, f).*

▶ **Definition 2.3** (Co-Kleisli Induction). Let $\mathbb{D} = (D, \varepsilon, \nu)$ be a comonad. Let (Y, k) be a D-algebra. A morphism $k^\ddagger : DX \to Y$ is called a *co-Kleisli-inductive solution* of *coequation* k, if and only if $k^\ddagger = k \circ^{\mathbf{Cl}(\mathbb{D})} k^\ddagger$.

▶ **Theorem 2.4** (Universality). *Given* any *D-algebra (Y, k) and co-Kleisli-inductive solution $k^\ddagger : DX \to Y$, there are canonical extensions to D-ca-homomorphisms $(k^\ddagger \circ f)$ from **all** $\mathbb{D}$-coalgebras (X, f).*

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3 Applications

▶ **Example 3.1** (Dynamical Systems & Time Series)**.** Consider the monad of pairing with an additive monoid Δ understood as time durations. Its algebras (S, g) are the monoid actions on state spaces, the most basic representation of dynamical systems. The coalgebras (I, f) of the underlying functor are loose specifications of discrete time series in terms of symbolic indices I and delays between samples. The ca-homomorphisms are the I-indexed time series over S that obey the specification. They exist necessarily if Δ is cancellative and the specification is free of time-like loops; the converse holds if Δ is a group.

▶ **Example 3.2** (Markov Chains & Stationary Distributions)**.** Consider the monad of countable discrete distributions. Its algebras are the convex sets, formal structures to reason about long-term behavior of stochastic processes. The coalgebras of the underlying functor are the Markov chains. The ca-homomorphisms can be understood as assignments of (symbolic or actual) stationary distributions to states. The sufficient condition for existence obtained by analogy to the above is the standard one, namely irreducibility.

▶ **Example 3.3** (Economic Games & Backwards Induction)**.** Consider the functor for multi-player perfect-information games from [1] and its cofree comonad. Its coalgebras are the vertex-labeled possibly infinite game trees. The algebras of the underlying functor are the inductive labelings. The ca-homomorphisms are labelings that are consistent under possibly non-wellfounded induction. The economic concept of backwards induction can be expressed in this framework concisely as an algebra invoking a single 'arg max' operator. The standard caveats regarding well-foundedness or auxiliary fixpoint constructions apply.

▶ **Example 3.4** (Lindenmayer Systems & Fractal Curves)**.** Consider the functor for L-systems with terminal symbols from [4] and its cofree comonad. Its coalgebras are the infinite derivation trees, indexed by start symbol. The algebras of the underlying functor are inductive interpretations of symbols. The ca-homomorpisms are interpretations of start symbols under generally non-well-founded induction. The mathematical fact that fractal curves can be specified by L-systems is expressed in this framework as an algebra of curves. It can be shown that this algebra is corecursive, that is L-systems specify unique curves, for a carefully constructed subcomonad capturing the geometric consistency conditions.

4 Conclusion

The framework presented here gives analogous and dual formal accounts for many previously disparate modes of reasoning about scientific models. We are confident that this is an appropriate level of abstraction for comparison and meta-reasoning, and that many more example instances and their relationships will follow.

References

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