A Fibrational Approach to Automata Theory
Eilenberg-type Correspondences in One

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Algebraic description of regular languages:

- Eilenberg’s variety theorem (Eilenberg, 1974),
- and a long list of variants
  - (Reutenauer, 1980)
  - (Pin, 1995)
  - ... (Polák, 2001)
  - (Straubing, 2002)
  - (Gehrke et al., 2009)
Motivation: a zoo of Eilenberg’s variety theorems

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- and a long list of variants
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  - ...
  - (Polák, 2001)
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Motivation: coalgebraic unification

Can we unify all of them?

😊 General (Local) Variety Theorem (Adámek et al., 2014 & 2015)

😔 Highly technical.

😢 Two independent arguments.

😢 An interesting instance (Straubing, 2002) is missing.

Goal

1. General Local Variety Theorem
   ➔ General Variety Theorem.

2. Cover all interesting instances.
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Goal

1. General Local Variety Theorem
   \[\rightarrow\] General Variety Theorem.

2. Cover all interesting instances.
Definition

A **variety of regular languages** is a set of regular languages closed under

- **Boolean ops.** $\cap$, $\cup$, $(-)^c$, $\emptyset$, $\Sigma^*$, $\Delta^*$, \ldots
- **Derivatives** $a^{-1}L = \{ w \in \Sigma^* \mid aw \in L \}$ and $La^{-1}$
- **Preimages** $f^{-1}(L)$ for any monoid homomorphism $\Delta^* \xrightarrow{f} \Sigma^*$.

Example

1. The variety of all regular languages.
2. The variety of star-free languages.
Background

**Definition**

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**Preimages** $f^{-1}(L)$ for any monoid homomorphism $\Delta^* \xrightarrow{f} \Sigma^*$.

**Example**

1. The variety of all regular languages.
2. The variety of star-free languages.
Pseudovarieties of monoids

Definition

A pseudovariety of monoids is a class of finite monoids closed under

1. finite products,
2. submonoids, and
3. quotients.

Example

1. The pseudovariety of all finite monoids.
2. The pseudovariety of aperiodic monoids.
A centerpiece of algebraic automata theory ...

Theorem (Eilenberg, 1974)

\[
\left( \text{varieties of regular languages} \right) \cong \left( \text{pseudovarieties of monoids} \right)
\]

And, this is not the only interesting class of regular languages.
Pin’s variety theorem

A positive variety is closed under \( \cap, \cup \), derivatives, and preimages.

Theorem (Pin, 1995)

\[
\left( \text{positive varieties of regular languages} \right) \supseteq \left( \text{pseudovarieties of ordered monoids} \right)
\]
Polák’s variety theorem

A disjunctive variety is closed under \( \cup \), derivatives, and preimages.

Theorem (Polák, 2001)

\[
\left( \text{disjunctive varieties of regular languages} \right) \cong \left( \text{pseudovarieties of idempotent semirings} \right)
\]
An xor variety is closed under symmetric differences $\oplus$, derivatives, and preimages.

**Theorem (Reutenauer, 1980)**

\[
\left( \begin{array}{c}
\text{xor varieties} \\
\text{of regular languages}
\end{array} \right) \sqsubseteq \left( \begin{array}{c}
pseudovarieties \text{ of} \\
\text{algebras over } \mathbb{Z}_2
\end{array} \right)
\]
A **local variety** over \( \Sigma \) is a class of languages \( L \subseteq \Sigma^* \) closed under \( \cup, \cap, (\neg)^C, \emptyset, \Sigma^* \) and derivatives.

A **local pseudovariety** over \( \Sigma \) is a class of \( M \leftarrow \Sigma^* \) closed under quotients and subdirect products.

**Theorem (Gehrke, Grigorieff and Pin, 2008)**

For each alphabet \( \Sigma \),

\[
\left( \text{local varieties of regular languages over } \Sigma \right) \cong \left( \text{local pseudovarieties of monoid over } \Sigma \right)
\]
Local Eilenberg theorems

1. A local variety over $\Sigma$ is a class of languages $L \subseteq \Sigma^*$ closed under $\cup$, $\cap$, $(\cdot)^C$, $\emptyset$, $\Sigma^*$ and derivatives.

2. A local pseudovariety over $\Sigma$ is a class of $M \hookrightarrow \Sigma^*$ closed under quotients and subdirect products.

Theorem (Gehrke, Grigorieff and Pin, 2008)

For each alphabet $\Sigma$,

\[
\begin{pmatrix}
\text{local varieties of} \\
\text{regular languages over } \Sigma
\end{pmatrix}
\cong
\begin{pmatrix}
\text{local pseudovarieties} \\
\text{of monoid over } \Sigma
\end{pmatrix}
\]
Let $\mathcal{C}$ and $\mathcal{D}$ be predual categories. General Local Variety Theorem:

**Theorem (Adámek, Milius, Myers, and Urbat, 2014)**

\[
\left( \text{local varieties of regular } \mathcal{C}\text{-languages over } \Sigma \right) \cong \left( \text{local pseudovarieties of } \mathcal{D}\text{-monoid over } \Sigma \right)
\]

General Variety Theorem:

**Theorem (Adámek, Milius, Myers, and Urbat, 2015)**

\[
\left( \text{varieties of regular } \mathcal{C}\text{-languages} \right) \cong \left( \text{pseudovarieties of } \mathcal{D}\text{-monoid} \right)
\]
Instances of General Local Variety Theorem

<table>
<thead>
<tr>
<th>$\mathcal{C}/\mathcal{D}$</th>
<th>local var. closed under</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Bool}/\text{Set}$</td>
<td>$\neg, \cap, \cup, \emptyset, \Sigma^*$</td>
</tr>
<tr>
<td>$\text{DistLat}/\text{Pos}$</td>
<td>$\cap, \cup, \emptyset, \Sigma^*$</td>
</tr>
<tr>
<td>$\lor\text{-SLat}/\lor\text{-SLat}$</td>
<td>$\cup, \emptyset$</td>
</tr>
<tr>
<td>$\mathbb{Z}_2\text{-Vec}/\mathbb{Z}_2\text{-Vec}$</td>
<td>$\oplus, \emptyset$</td>
</tr>
<tr>
<td>$\text{BR}/\text{Set}^*$</td>
<td>$\oplus, \cup, \emptyset$</td>
</tr>
</tbody>
</table>

Organising local varieties as an opfibration to get non-local correspondences.
An opfibration of local varieties, informally

1. \( f_*(V) \) is the “largest” local variety closed under \( f \)-preimages.
2. \( p \) is equivalent to a functor \( \text{Free}(\text{Mon} \mathcal{D}) \to \text{Pos} \).
An opfibration of local varieties, informally

1. \( f_*(V) \) is the “largest” local variety closed under \( f \)-preimages.

2. \( p \) is equivalent to a functor \( \text{Free}(\text{Mon}D) \to \text{Pos} \).
**Definition**

The category $\textbf{LAN}$ consists of

- **objects** $(\Sigma, V)$, a local variety $V$ of regular languages of $\Sigma$;
- **morphisms** $(\Sigma, V) \xrightarrow{f} (\Delta, W)$, a morphism $\Sigma^* \xrightarrow{f} \Delta^*$ s.t. $V$ is closed under $f$-preimages

\[
\begin{array}{ccc}
W & \xrightarrow{\sim} & V \\
\downarrow & & \downarrow \\
\text{Reg}(\Delta) & \xrightarrow{f^{-1}} & \text{Reg}(\Sigma)
\end{array}
\]

with a projection $p: \textbf{LAN} \rightarrow \textbf{Free}(\text{Mon}\mathcal{D})$. 
The category $\mathbf{LPV}$ consists of

objects $(\Sigma, P)$, a local pseudovariety $V$ of monoids over $\Sigma$

morphisms $(\Sigma, P) \xrightarrow{f} (\Delta, Q)$, a monoid morphism $f$ such that:

$$
\begin{array}{ccc}
\Psi \Sigma^* & \xrightarrow{f} & \Psi \Delta^* \\
\downarrow \exists e_M \in P & \quad & \downarrow \forall e_N \in Q \\
M & \rightarrow & N
\end{array}
$$

with a projection $p: \mathbf{LPV} \rightarrow \text{Free}(\text{Mon}\mathcal{D})$. 
An opfibrations of local varieties and related structures

**LAN** the opfibration of local varieties of languages in $C$.

**LPV** the opfibration of local pseudovarieties of $\mathcal{D}$-monoids.

**PFMon** the opfibration of finitely generated profinite $\mathcal{D}$-monoids.
An opfibrations of local varieties and related structures

\[ \text{FLan} \quad \text{the opfibration of local varieties of languages in} \ C. \]

\[ \text{LPV} \quad \text{the opfibration of local pseudovarieties of} \ D\text{-monoids.} \]

\[ \text{PFMon} \quad \text{the opfibration of finitely generated profinite} \ D\text{-monoids}. \]
The connection between local and global

\[ \text{LAN} \cong \text{LPV} \]

\[ \text{Free(Mon}\mathcal{D}) \]

**Theorem (Fibrational Variety Isomorphism)**

*Opfibrations LAN and LPV are isomorphic.*
The key observation

\[ \text{LAN} \xrightarrow{\cong} \text{LPV} \xrightarrow{\cong} \text{PFMon} \]

Free(Mon\(\mathcal{D}\))

Global sections of

**LAN** varieties of regular languages in \(\mathcal{C}\)

**PFMon** profinite equational theories of \(\mathcal{D}\)-monoids

Corollary

\[ \text{varieties of regular } \mathcal{C}\text{-languages} \xrightarrow{\cong} \text{profinite equational theories of } \mathcal{D}\text{-monoid} \]
The key observation

\[
\text{LAN} \xrightarrow{\mathcal{R}} \text{LPV} \xrightarrow{\mathcal{R}} \text{PFMon}
\]

Free(Mon\(\mathcal{D}\))

Global sections of

**LAN** varieties of regular languages in \(C\)

**PFMon** profinite equational theories of \(\mathcal{D}\)-monoids

**Corollary**

\[
\begin{pmatrix}
\text{varieties of regular } C\text{-languages} \\
\end{pmatrix}
\mathcal{R}
\begin{pmatrix}
\text{profinite equational theories of } \mathcal{D}\text{-monoid}
\end{pmatrix}
\]
Correspondence between pseudovarieties and profinite equations

Modification of (Reiterman, 1982) & (Banaschewski, 1983):

Theorem

\[
\begin{pmatrix}
(\text{profinite equational theories of } D\text{-monoid}) \\
(\text{theories of } D\text{-monoid})
\end{pmatrix}
\cong
\begin{pmatrix}
(\text{pseudovarieties}) \\
(\text{of } D\text{-monoids})
\end{pmatrix}
\]
General Variety Theorem as a corollary

**Corollary**

\[
\left( \text{varieties of regular } C\text{-languages} \right) \cong \left( \text{pseudovarieties of } D\text{-monoids} \right)
\]

**Proof.**

By Fibrational Isomorphism and Reiterman’s Correspondence. \(\square\)
Change of base, for free!

For each subcategory $S$, take the pullback along the inclusion

$$
\begin{align*}
\text{LAN}_C & \leftarrow \text{LAN} \\
\downarrow p_C & \downarrow p \\
S & \xrightarrow{j} \text{Free}(\text{MonD})
\end{align*}
\quad
\begin{align*}
\text{PFMon}_C & \leftarrow \text{PFMon} \\
\downarrow q'_C & \downarrow q' \\
S & \xrightarrow{j} \text{Free}(\text{MonD})
\end{align*}
$$

Corollary

$$
\left( \begin{array}{c} S\text{-varieties of} \\ \text{regular } C \text{-languages} \end{array} \right) \mathrel{\overset{\mathcal{R}}{\subseteq}} \left( \begin{array}{c} \text{profinite equational} \\ S\text{-theories of } D \text{-monoids} \end{array} \right)
$$

The missing case (Straubing, 2002) is a special instance when

1. $C/D = \text{Set/BA}$ and
2. $S$ is a non-full subcategory on all free $D$-monoids.
Change of base, for free!

For each subcategory $S$, take the pullback along the inclusion

$$\begin{align*}
\text{LAN}_C & \xleftarrow{p_C} \text{LAN} \\
S & \xrightarrow{j} \text{Free}(\text{Mon}_D)
\end{align*}$$

$$\begin{align*}
\text{PFMon}_C & \xleftarrow{q'_C} \text{PFMon} \\
S & \xrightarrow{j} \text{Free}(\text{Mon}_D)
\end{align*}$$

Corollary

$$\left( \begin{array}{c}
S\text{-varieties of regular } C\text{-languages} \\
\end{array} \right) \cong \left( \begin{array}{c}
\text{profinite equational } S\text{-theories of } D\text{-monoids}
\end{array} \right)$$

The missing case (Straubing, 2002) is a special instance when

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2. $S$ is a non-full subcategory on all free $D$-monoids.
Conclusion

General Local Variety Theorem $\implies$ General Variety Theorem.

Varieties of regular languages are dual to profinite equational theories.

Eilenberg’s variety theorem $=$ Reiterman’s theorem $+$ duality.

What is categorical Reiterman’s theorem ...

A 2-duality between pseudovarieties and “profinite monads”?

Thank you for your attention.
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