A Fibrational Approach to Automata Theory

Eilenberg-type Correspondences in One

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Motivation: a zoo of Eilenberg's variety theorems

Algebraic description of regular languages:

☺ Eilenberg's variety theorem (Eilenberg, 1974),

and a long list of variants

☺ (Reutenauer, 1980)

☺ (Pin, 1995)

😐 ...

😐 (Polák, 2001)

😐 (Straubing, 2002)

😐 (Gehrke et al., 2009)

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Motivation: coalgebraic unification

Can we unify all of them?

- ☺ General (Local) Variety Theorem (Adámek et al., 2014 & 2015)
- ☹ Highly technical.
- ☹ Two independent arguments.
- ☹ An interesting instance (Straubing, 2002) is missing.

Goal

1. General Local Variety Theorem
   $\Longrightarrow$ General Variety Theorem.
2. Cover all interesting instances.

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Background

Definition

A variety of regular languages is a set of regular languages closed under

Boolean ops. $\cap$, $\cup$, $(-)^{\complement}$, $\emptyset$, $\Sigma^*, \Delta^*, \ldots$

Derivatives $a^{-1}L = \{\, w \in \Sigma^* \mid aw \in L \,\}$ and La^{-1}

Preimages $f^{-1}(L)$ for any monoid homomorphism $\Delta^* \xrightarrow{f} \Sigma^*$.

Example

1. The variety of all regular languages.
2. The variety of star-free languages.

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Pseudovarieties of monoids

Definition

A pseudovariety of monoids is a class of *finite* monoids closed under

1. finite products,
2. submonoids, and
3. quotients.

Example

1. The pseudovariety of all finite monoids.
2. The pseudovariety of aperiodic monoids.

A centerpiece of algebraic automata theory ...

Theorem (Eilenberg, 1974)

$$\begin{pmatrix} \textit{varieties} \\ \textit{of regular languages} \end{pmatrix} \cong \begin{pmatrix} \textit{pseudovarieties of} \\ \textit{monoids} \end{pmatrix}$$

And, this is not the only interesting class of regular languages.

Pin's variety theorem

A positive variety is closed under $\cap, \cup$, derivatives, and preimages.

Theorem (Pin, 1995)

$$\begin{pmatrix} \textit{positive varieties} \\ \textit{of regular languages} \end{pmatrix} \cong \begin{pmatrix} \textit{pseudovarieties of} \\ \textit{ordered monoids} \end{pmatrix}$$

Polák's variety theorem

A disjunctive variety is closed under $\cup$, derivatives, and preimages.

Theorem (Polák, 2001)

$$\begin{pmatrix} \textit{disjunctive varieties} \\ \textit{of regular languages} \end{pmatrix} \cong \begin{pmatrix} \textit{pseudovarieties of} \\ \textit{idempotent semirings} \end{pmatrix}$$

Reutenauer's variety theorem

An *xor* variety is closed under symmetric differences $\oplus$, derivatives, and preimages.

Theorem (Reutenauer, 1980)

$$\begin{pmatrix} \textit{xor varieties} \\ \textit{of regular languages} \end{pmatrix} \cong \begin{pmatrix} \textit{pseudovarieties of} \\ \textit{algebras over } \mathbb{Z}_2 \end{pmatrix}$$

Local Eilenberg theorems

1. A local variety over Σ is a class of languages $L \subseteq \Sigma*$ closed under $\cup, \cap, (-)^{\complement}, \emptyset, \Sigma^*$ and derivatives.
2. A local pseudovariety over Σ is a class of $M \twoheadleftarrow \Sigma^*$ closed under quotients and subdirect products.

Theorem (Gehrke, Grigorieff and Pin, 2008)

For each alphabet Σ,

$$\begin{pmatrix} \textit{local varieties of} \\ \textit{regular languages over } \Sigma \end{pmatrix} \cong \begin{pmatrix} \textit{local pseudovarieties} \\ \textit{of monoid over } \Sigma \end{pmatrix}$$

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Category theorists: *Veni, vidi, vici*

Let $\mathcal{C}$ and $\mathcal{D}$ be predual categories. General Local Variety Theorem:

Theorem (Adámek, Milius, Myers, and Urbat, 2014)

$$\begin{pmatrix} \textit{local varieties of regular} \\ \mathcal{C}\textit{-languages over } \Sigma \end{pmatrix} \cong \begin{pmatrix} \textit{local pseudovarieties} \\ \textit{of } \mathcal{D}\textit{-monoid over } \Sigma \end{pmatrix}$$

General Variety Theorem:

Theorem (Adámek, Milius, Myers, and Urbat, 2015)

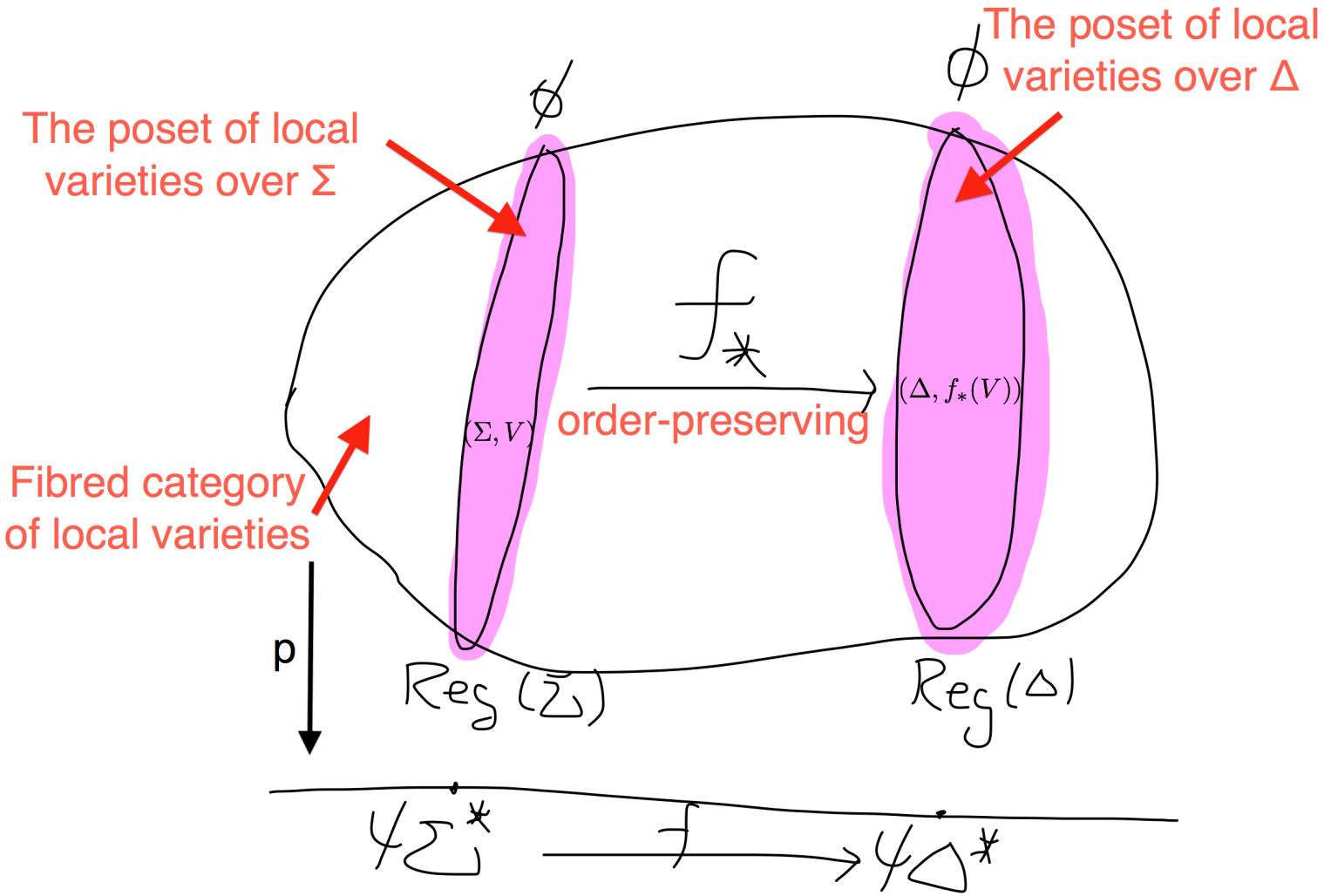
$$\begin{pmatrix} \textit{varieties of} \\ \textit{regular } \mathcal{C}\textit{-languages} \end{pmatrix} \cong \begin{pmatrix} \textit{pseudovarieties} \\ \textit{of } \mathcal{D}\textit{-monoid} \end{pmatrix}$$

Instances of General Local Variety Theorem

$\mathcal{C}/\mathcal{D}$	local var. closed under
Bool/Set	$\neg, \cap, \cup, \emptyset, \Sigma^*$
DistLat/Pos	$\cap, \cup, \emptyset, \Sigma^*$
$\vee$-**SLat**/$\vee$-**SLat**	$\cup, \emptyset$
$\mathbb{Z}_2$-**Vec**/$\mathbb{Z}_2$-**Vec**	$\oplus, \quad \emptyset$
BR/$\mathbf{Set}_*$	$\oplus, \quad \cup, \emptyset$

Organising local varieties as an opfibration to get non-local correspondences.

An opfibration of local varieties, informally



1. $f_*(V)$ is the "largest" local variety closed under f-preimages.
2. p is equivalent to a functor $\mathbf{Free}(\mathbf{Mon}\mathcal{D}) \to \mathbf{Pos}$.

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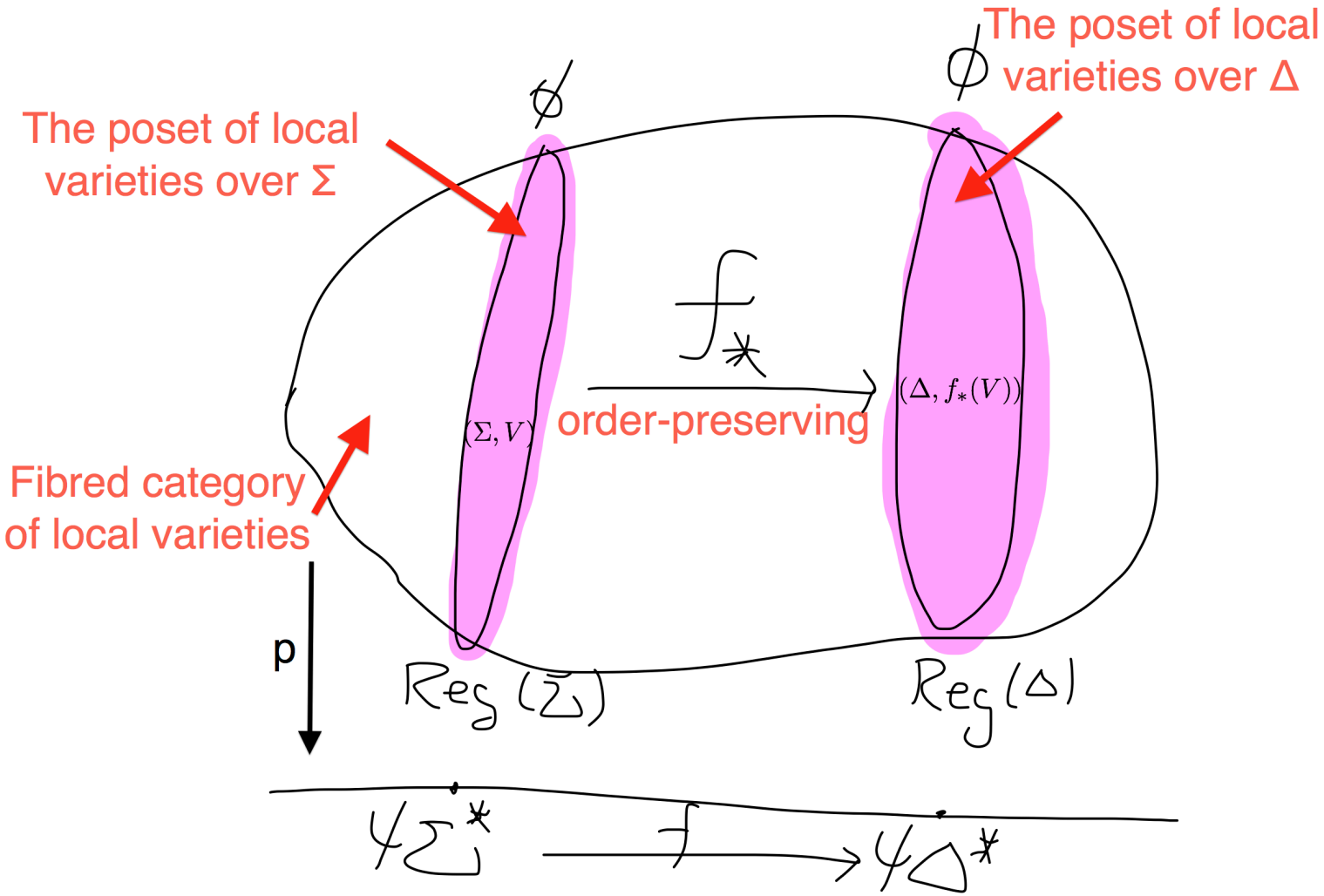

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An opfibration of local varieties of regular languages

Definition

The category $\mathbf{LAN}$ consists of

objects (Σ, V), a local variety V of regular languages of Σ;

morphisms $(\Sigma, V) \xrightarrow{f} (\Delta, W)$, a morphism $\Sigma^* \xrightarrow{f} \Delta^*$ s.t. V is closed under f-preimages

$$\begin{tikzcd}
W \arrow[r, dashed] \arrow[d, tail] & V \arrow[d, tail] \\
\mathrm{Reg}(\Delta) \arrow[r, "f^{-1}"'] & \mathrm{Reg}(\Sigma)
\end{tikzcd}$$

with a projection $p\colon \mathbf{LAN} \to \mathbf{Free}(\mathbf{Mon}\mathcal{D})$.

Opfibrations of local pseudovarieties of monoids

Definition

The category $\mathbf{LPV}$ consists of

objects (Σ, P), a local pseudovariety V of monoids over Σ

morphisms $(\Sigma, P) \xrightarrow{f} (\Delta, Q)$, a monoid morphism f such that:

$$\begin{tikzcd}
\Psi\Sigma^* \arrow[r, "f"] \arrow[d, two heads, "\exists\, e_M \in P"'] & \Psi\Delta^* \arrow[d, two heads, "\forall\, e_N \in Q"] \\
M \arrow[r, dashed] & N
\end{tikzcd}$$

with a projection $p\colon \mathbf{LPV} \to \mathbf{Free}(\mathbf{Mon}\mathcal{D})$.

An opfibrations of local varieties and related structures

$$\begin{tikzcd}
\mathbf{LAN} \arrow[dr, "p"'] & \mathbf{LPV} \arrow[d, "q"] \\
& \mathbf{Free}(\mathbf{Mon}\mathcal{D})
\end{tikzcd}$$

FLan the opfibration of local varieties of languages in $\mathcal{C}$.

LPV the opfibration of local pseudovarieties of $\mathcal{D}$-monoids.

PFMon the opfibration of finitely generated profinite $\mathcal{D}$-monoids .

An opfibrations of local varieties and related structures

$$\begin{tikzcd}
\mathbf{LPV} \arrow[rr, "\cong", "\mathsf{Lim}"'] \arrow[d, "q"] & & \mathbf{PFMon} \arrow[dll, "q'"] \\
\mathbf{Free}(\mathbf{Mon}\mathcal{D}) & &
\end{tikzcd}$$

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The connection between local and global

$$\begin{tikzcd}
\mathbf{LAN} \arrow[r, "\cong"] \arrow[dr, "p"'] & \mathbf{LPV} \arrow[d, "q"] \\
& \mathbf{Free}(\mathbf{Mon}\mathcal{D})
\end{tikzcd}$$

Theorem (Fibrational Variety Isomorphism)

Opfibrations $\mathbf{LAN}$ *and* $\mathbf{LPV}$ *are isomorphic.*

The key observation

$$\begin{tikzcd}
\mathbf{LAN} \arrow[r, "\cong"] & \mathbf{LPV} \arrow[r, "\cong"] & \mathbf{PFMon} \\
& \mathbf{Free}(\mathbf{Mon}\mathcal{D}) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Global sections of

LAN varieties of regular languages in $\mathcal{C}$

PFMon profinite equational theories of $\mathcal{D}$-monoids

Corollary

$$\begin{pmatrix} \textit{varieties of} \\ \textit{regular } \mathcal{C}\textit{-languages} \end{pmatrix} \cong \begin{pmatrix} \textit{profinite equational} \\ \textit{theories of } \mathcal{D}\textit{-monoid} \end{pmatrix}$$

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Correspondence between pseudovarieties and profinite equations

Modification of (Reiterman, 1982) & (Banaschewski, 1983):

Theorem

$$\begin{pmatrix} \textit{profinite equational} \\ \textit{theories of } \mathcal{D}\textit{-monoid} \end{pmatrix} \cong \begin{pmatrix} \textit{pseudovarieties} \\ \textit{of } \mathcal{D}\textit{-monoids} \end{pmatrix}$$

General Variety Theorem as a corollary

Corollary

$$\begin{pmatrix} \textit{varieties of regular} \\ \mathcal{C}\textit{-languages} \end{pmatrix} \cong \begin{pmatrix} \textit{pseudovarieties} \\ \textit{of } \mathcal{D}\textit{-monoids} \end{pmatrix}$$

Proof.

By Fibrational Isomorphism and Reiterman's Correspondence. $\square$

Change of base, for free!

For each subcategory S, take the pullback along the inclusion

$$\begin{tikzcd} \mathbf{LAN}_{\mathsf{C}} \arrow[r, hook] \arrow[d, "p_{\mathsf{C}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{LAN} \arrow[d, "p"] \\ \mathsf{S} \arrow[r, hook, "j"'] & \mathbf{Free}(\mathbf{Mon}\mathcal{D}) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{PFMon}_{\mathsf{C}} \arrow[r, hook] \arrow[d, "q'_{\mathsf{C}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{PFMon} \arrow[d, "q'"] \\ \mathsf{S} \arrow[r, hook, "j"'] & \mathbf{Free}(\mathbf{Mon}\mathcal{D}) \end{tikzcd}$$

Corollary

$$\begin{pmatrix} S\text{-varieties of} \\ \text{regular } \mathcal{C}\text{-languages} \end{pmatrix} \cong \begin{pmatrix} \text{profinite equational} \\ S\text{-theories of } \mathcal{D}\text{-monoids} \end{pmatrix}$$

The missing case (Straubing, 2002) is a special instance when

1. $\mathcal{C}/\mathcal{D} = \mathbf{Set}/\mathbf{BA}$ and
2. S is a non-full subcategory on all free $\mathcal{D}$-monoids.

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- ☺ Genearl Local Vareity Theorem $\Longrightarrow$ General Variety Theorem.
- ☺ Varieties of regular languages are dual to profinite equational theories.
- ☺ Eilenberg's variety theorem = Reiterman's theorem + duality.
- ☹ What is categorical Reiterman's theorem ...
  A 2-duality between pseudovarieties and "profinite monads"?

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