On coalgebras over algebras

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10th International Workshop on Coalgebraic Methods in Computer Science

Outline

Motivation

- Starting data: category $\mathcal{C}$, endofunctor $H : \mathcal{C} \longrightarrow \mathcal{C}$
- Among fixed points: final coalgebra, initial algebra
- Categories enriched over complete metric spaces: unique fixed point [Adamek, Reiterman 1994]
- Categories enriched over cpo: final coalgebra L coincides with initial algebra I [Plotkin, Smyth 1983]

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- Category with no extra structure *Set*: final coalgebra L is completion of initial algebra I [Barr 1993]
  Deficit: if $H0 = 0$, important cases not covered (as $A \times (-)^n$, $\mathcal{D}$, $\mathcal{P}_{\kappa+}$)
- Locally finitely presentable categories: $Hom(B, L)$ completion of $Hom(B, I)$ for all finitely presentable objects B [Adamek 2003]

In this talk

- Category: $Alg(\mathbf{M})$ for a Set-monad $\mathbf{M}$
- $Alg(\mathbf{M})$-functor: obtained from lifting

Outline

Construction of the final coalgebra

- Assumption 1: functor $H : Set \longrightarrow Set$ ω^{op}-continuous
- Terminal sequence

$$1 \xleftarrow{t} H1 \longleftarrow \cdots \longleftarrow H^n 1 \xleftarrow{H^n t} \cdots$$

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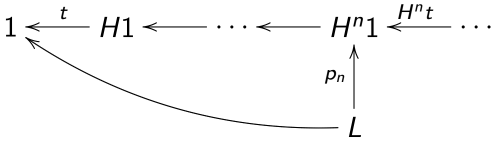

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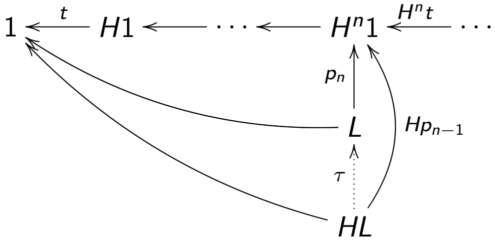
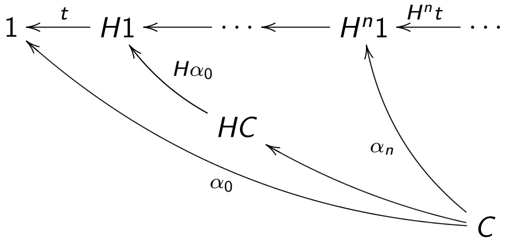


- The limit of the terminal sequence is the final H-coalgebra by cocontinuity $\xi = \tau^{-1} : L \simeq HL$

Final coalgebras and anamorphisms

- For each coalgebra $C \xrightarrow{\xi_C} HC$ there is a cone over the terminal sequence



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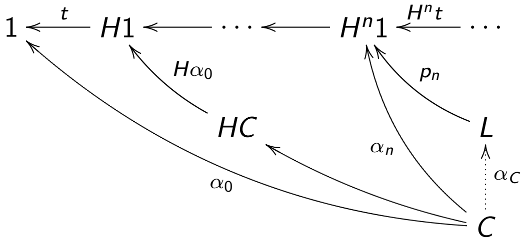

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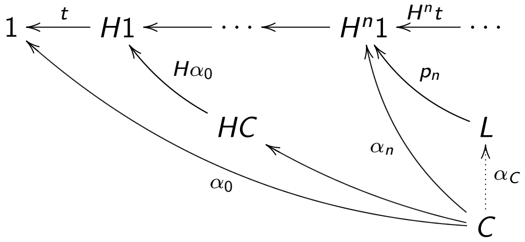

- Topology:
  Discrete topology on H^n1.
  Initial topology on L, HL and $C \Longrightarrow L$ complete ultrametric space.
  All maps are continuous.

Outline

Lifting functors to algebras over a monad

- Monad $\mathbf{M} = (M, m : M^2 \longrightarrow M, u : Id \longrightarrow M)$
- Adjunction $F^{\mathbf{M}} \dashv U^{\mathbf{M}} : Alg(\mathbf{M}) \longrightarrow Set$
- Initial object $M^2 0 \longrightarrow M0$, terminal object $M1 \longrightarrow 1$

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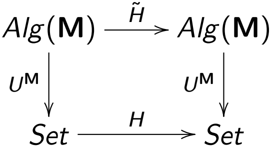
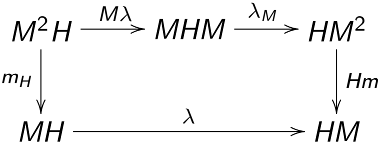
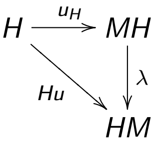
Lifting of H to $Alg(\mathbf{M})$ $\iff$ Distributive law $\lambda : MH \longrightarrow HM$







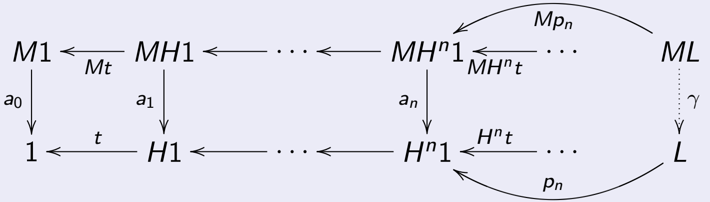
The final coalgebra and the lifting

Assumption 2: there is a lifting $\widetilde{H}$ of H to $Alg(\mathbf{M})$

Then $(L, L \xrightarrow{\xi} HL)$ inherits an algebra structure map $ML \xrightarrow{\gamma} L$ making it the final $\widetilde{H}$-coalgebra.

Lemma

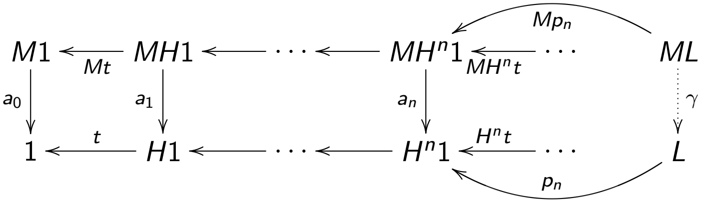
The cone $ML \xrightarrow{Mp_n} MH^n1 \xrightarrow{a_n} H^n1$ is induced by the H-coalgebra structure of ML



Hence the unique coalgebra map $\gamma : ML \longrightarrow L$ is also the anamorphism $\alpha_{ML} : ML \longrightarrow L$ for the coalgebra ML.

The final coalgebra and the lifting

- Diagram in $Alg(\mathbf{M})$ with limiting lower sequence



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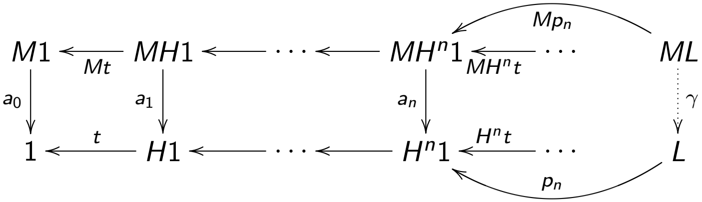

- Topology

  Discrete topology on both sequences

  Initial topologies on ML and L

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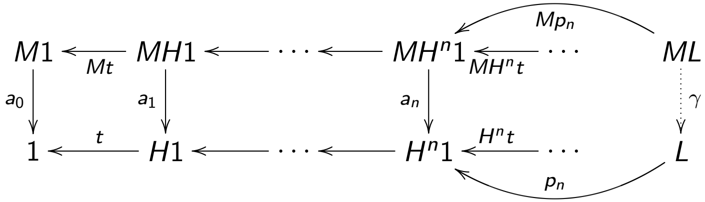

- Topology
  Discrete topology on both sequences
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Proposition

The final H-coalgebra inherits a structure of a topological $\mathbf{M}$*-algebra.*

Fixed points of lifted functor

- Initial-terminal $\widetilde{H}$-sequences:

$$
\begin{array}{ccccccccc}
M0 & \longrightarrow & HM0 & \longrightarrow & \cdots & \longrightarrow & H^n M0 & \longrightarrow & \cdots \\
\downarrow{\scriptstyle s} & & \downarrow{\scriptstyle Hs} & & & & \downarrow{\scriptstyle H^n s} & & \\
1 & \underset{t}{\longleftarrow} & H1 & \longleftarrow & \cdots & \longleftarrow & H^n 1 & \longleftarrow & \cdots
\end{array}
$$

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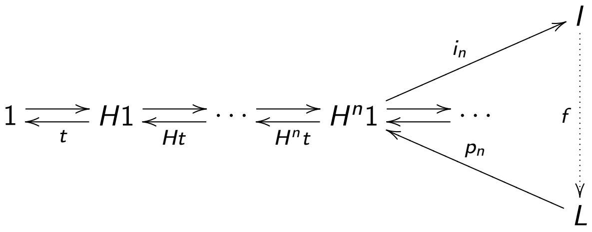
Fixed points of lifted functor

- [Adamek 2003] $\widetilde{H}$ has also (non empty) initial algebra I built upon this sequence in $Alg(\mathbf{M})$, with unique **M**-algebra monomorphism $f : I \longrightarrow L$



Main result

Theorem

Let H be a Set-endofunctor ω^{op}-continuous and $\mathbf{M}$ a monad on Set such that:

1. *H admits a lifting $\tilde{H}$ to $Alg(\mathbf{M})$*
2. *$M0 = 1$ in $Alg(\mathbf{M})$*

Then the final H-coalgebra is the completion of the initial $\widetilde{H}$-algebra under a suitable (ultra)metric.

Idea of the proof...

Take on I the coarsest topology such that f is continuous

$$\begin{array}{c} I \\ \Big\downarrow{\scriptstyle f} \\ L \end{array}$$

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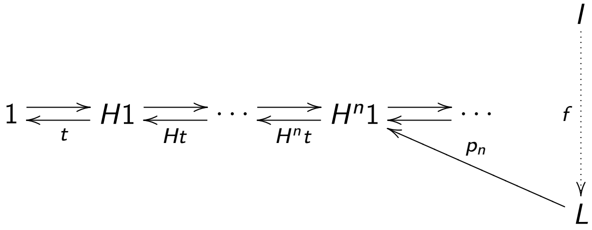
Take on I the coarsest topology such that f is continuous = initial topology from the cone $p_n \circ f$



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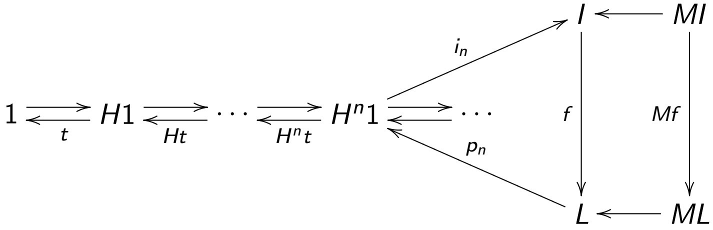

Obtain $MI \longrightarrow I$ topological algebra.

Density of initial algebra into the final coalgebra

Remember L is complete ultrametric space.
Then the image of I is dense in L:

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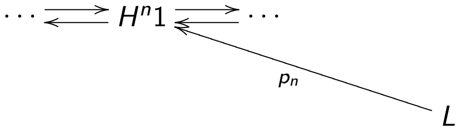

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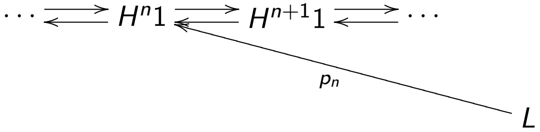

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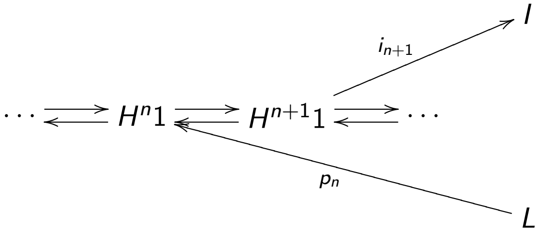

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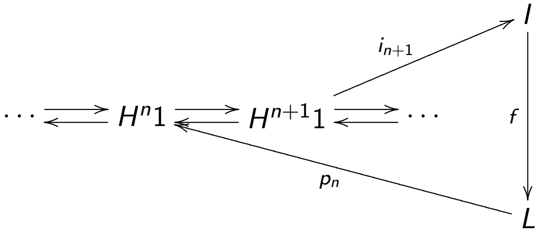

An example

- Consider $HX = \mathbb{k} \times X^A$.
  - Coalgebras are Moore automata.
  - Final coalgebra is $\mathbb{k}^{A^*}$, initial algebra is empty.
  - For any monad **M**, such that $\mathbb{k}$ carries an **M**-algebra structure, a lifting $\widetilde{H}$ always exists.
  - Hence the theorem applies: $\mathbb{k}^{A^*}$ as the completion of initial $\widetilde{H}$-algebra.

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  - Consider the monad $\mathbf{M} = (M, m, u)$ given by $MX = \{f : X \longrightarrow \mathbb{k} | supp(f) \text{ finite}\}$

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  - Consider the monad $\mathbf{M} = (M, m, u)$ given by $MX = \{f : X \longrightarrow \mathbb{k} | supp(f) \text{ finite}\}$
  - Final H-coalgebra: $\mathbb{k}\langle\langle A \rangle\rangle$
  - Initial $\widetilde{H}$-algebra: $\mathbb{k}\langle A \rangle$

Outline

Lifting functors to categories of algebras

Lifting of H to $Alg(\mathbf{M})$ $\iff$ Distributive law $\lambda : MH \longrightarrow HM$

Lifting functors to categories of algebras

Lifting of H to $Alg(\mathbf{M})$	$\Longleftrightarrow$	Distributive law $\lambda : MH \longrightarrow HM$
Lifting of T to $Kl(\mathbf{M})$	$\Longleftrightarrow$	Distributive law $\varsigma : TM \longrightarrow MT$

$$\begin{array}{ccc} Kl(\mathbf{M}) & \xrightarrow{\hat{T}} & Kl(\mathbf{M}) \\ \uparrow{\scriptstyle F_{\mathbf{M}}} & & \uparrow{\scriptstyle F_{\mathbf{M}}} \\ Set & \xrightarrow{T} & Set \end{array}$$

$$\begin{array}{ccccc} TM^2 & \xrightarrow{\varsigma_M} & MTM & \xrightarrow{M\varsigma} & M^2T \\ \downarrow{\scriptstyle Tm} & & & & \downarrow{\scriptstyle m_T} \\ TM & & \xrightarrow{\varsigma} & & MT \end{array}$$

$$\begin{array}{ccc} T & \xrightarrow{Tu} & TM \\ & \searrow{\scriptstyle u_T} & \downarrow{\scriptstyle \varsigma} \\ & & MT \end{array}$$

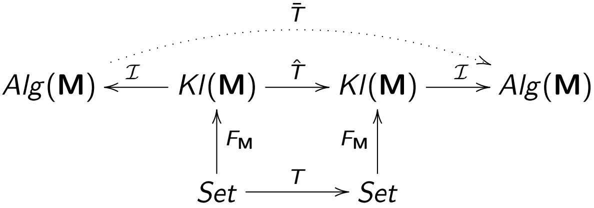
More on Kleisli lift

- Assume Kleisli lift of T exists

$$\begin{array}{ccc} Kl(\mathbf{M}) & \xrightarrow{\hat{T}} & Kl(\mathbf{M}) \\ \uparrow F_{\mathbf{M}} & & F_{\mathbf{M}} \uparrow \\ Set & \xrightarrow{T} & Set \end{array}$$

More on Kleisli lift

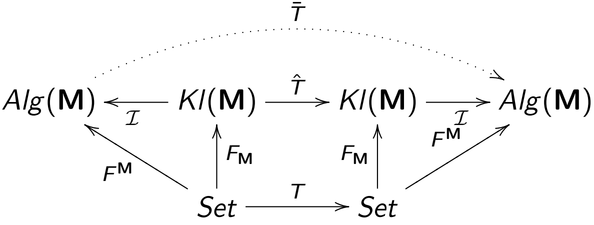
- Assume Kleisli lift of T exists
- Consider also $\mathcal{I} : Kl(\mathbf{M}) \longrightarrow Alg(\mathbf{M})$



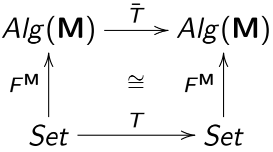
- Construct the left Kan extension $\bar{T} = Lan_{\mathcal{I}}(\mathcal{I}\hat{T})$

More on Kleisli lift

- Upper diagram commutes: $\mathcal{I}\hat{T} \cong \bar{T}\mathcal{I}$.



- It follows that



Commuting pair of *Set*-endofunctors

- Take two functors T, H on Set such that:
  - H has a lift $\widetilde{H}$ to $Alg(\mathbf{M})$
  - T has a lift $\hat{T}$ to $Kl(\mathbf{M})$, hence an extension $\bar{T}$ to $Alg(\mathbf{M})$
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- Then

$$MT = U^{\mathbf{M}} F^{\mathbf{M}} T \cong U^{\mathbf{M}} \bar{T} F^{\mathbf{M}} \cong U^{\mathbf{M}} \widetilde{H} F^{\mathbf{M}} = H U^{\mathbf{M}} F^{\mathbf{M}} = HM$$

- Hence $MT \cong HM$

Commuting pair of *Set*-endofunctors

Definition

Let $\mathbf{M} = (M, m, u)$ be a monad on *Set*. A pair of *Set*-endofunctors (T, H) such that $MT \cong HM$ is called an **M**-*commuting pair*.

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Trivial examples:

- $T = H = Id$ or $T = H = M$, $\mathbf{M}$ any monad
- $T = H = A + (-)$, $\mathbf{M} = B + (-)$
- $T = H = A \times (-)$, $\mathbf{M} = B \times (-)$
- $\mathbf{M}$ idempotent monad, $H = M$, $T = Id$ or $H = Id$, $T = M$

Commuting pair of *Set*-endofunctors

- $\widetilde{H} \cong \bar{T}$ implies not only natural isomorphism $MT \cong HM$, but also isomorphism of algebras

$$\begin{array}{ccccc} MHMX & \xrightarrow{\lambda_{MX}} & HM^2X & \xrightarrow{Hm_X} & HMX \\ \downarrow{\cong} & & & & \downarrow{\cong} \\ M^2TX & & \xrightarrow[m_{TX}]{} & & MTX \end{array}$$

because of $\widetilde{H}F^{\mathbf{M}} \cong \bar{T}F^{\mathbf{M}} \cong F^{\mathbf{M}}T$

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- If the algebra lift of H is isomorphic to the algebra extension of T, then H and T form a commuting pair by an algebra isomorphism $HM \cong MT$.

Commuting pair of *Set*-endofunctors

- Conversely, assume a commuting pair (T, H) such that corresponding lifts exists, and $HMX \cong MTX$ as algebras.
- This implies $\tilde{H} \cong \bar{T}$ on free algebras.

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- Assume M, T, H finitary. Then $\bar{T}$ is determined by its action on finitely free algebras, and so is $\widetilde{H}$ (because it preserves sifted colimits)
- Obtain $\tilde{H} \cong \bar{T}$

Commuting pair and algebra lift-extension isomorphism

Theorem

Let H, T two endofunctors and $\mathbf{M}$ a monad on Set, such that H and T have algebra lift $\widetilde{H}$, respectively Kleisli lift with respect to the monad $\mathbf{M}$, with $\bar{T}$ the corresponding left Kan extension to algebras. Then:

- *If $\widetilde{H} \cong \bar{T}$, then (T, H) form an $\mathbf{M}$-commuting pair and $HMX \cong MTX$ as algebras for any X.*
- *If M, H, T are finitary and $MT \cong HM$ as algebras, then $\widetilde{H} \cong \bar{T}$.*

Commuting pair and algebra lift-extension isomorphism

Corollary

Let H, T two endofunctors and $\mathbf{M}$ a monad on Set, such that:

- *M, H, T are finitary*
- *H is ω^{op}-continuous*
- *H has algebra lift, T has Kleisli lift*
- *$MT \cong HM$ as algebras*
- *$M0 = 1$ as algebras*

Then the final H-coalgebra is the completion of the free $\mathbf{M}$-algebra built on the initial T-algebra.

An example

- Consider $TX = 1 + A \cdot X$ and **M** any monad.
- Kleisli lift exists
- Algebra extension $\bar{T}X = F^{\mathbf{M}}1 + A \cdot X$

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- Assume $Alg(\mathbf{M})$ has biproducts. Then $\bar{T}$ is the lifting to $Alg(\mathbf{M})$ of the *Set*-endofunctor $HX = M1 \times X^A$.

An example

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- Hence (T, H) form a commuting pair.

More on commuting pairs

Given T and H, find the linking monad such that (T, H) form a commuting pair.

More on commuting pairs

Given T and H, find the linking monad such that (T, H) form a commuting pair.
Given monad $\mathbf{M}$ and (T, H) commuting pair, find both distributive laws.

More on commuting pairs

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Particular case: T contains products, as in $T_1X = A \times X$ or $T_2X = X \times X$
Then $\bar{T}_1X = F^{\mathbf{M}}A \otimes X$, respectively $\bar{T}_2X = X \otimes X$ (as $F^{\mathbf{M}}$ is monoidal)

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Then $\bar{T}_1X = F^{\mathbf{M}}A \otimes X$, respectively $\bar{T}_2X = X \otimes X$ (as $F^{\mathbf{M}}$ is monoidal) If $\mathbf{M}$ is finitary, then $U^{\mathbf{M}}$ sends $(X, x) \otimes (Y, y)$ to the reflexive *Set*-coequalizer of

$$M(MX \times MY) \underset{m_{X\times Y} \circ M\varphi_2}{\overset{M(x\times y)}{\rightrightarrows}} M(X \times Y)$$

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M commutative monad, T analytic functor $\Longrightarrow$ distributive law exists
Particular case: T contains products, as in $T_1X = A \times X$ or $T_2X = X \times X$
Then $\bar{T}_1X = F^{\mathbf{M}}A \otimes X$, respectively $\bar{T}_2X = X \otimes X$ (as $F^{\mathbf{M}}$ is monoidal) If **M** is finitary, then $U^{\mathbf{M}}$ sends $(X, x) \otimes (Y, y)$ to the reflexive *Set*-coequalizer of

$$M(MX \times MY) \underset{m_{X \times Y} \circ M\varphi_2}{\overset{M(x \times y)}{\rightrightarrows}} M(X \times Y)$$

Hence for any such T and **M**, a corresponding commuting pair (T, H) can be constructed.

More on commuting pairs

The algebra lift

More complicated, even for simplest cases of polynomial functors:

More on commuting pairs

The algebra lift

More complicated, even for simplest cases of polynomial functors:

- H constant functor, then the image of H must be carrier of an **M**-algebra
- $HX = A \times X^n$, then $\exists$ lifting $\Longrightarrow$ A is the carrier of an **M**-algebra
- $HX = A + X$ or $HX = X + X$, there is no obvious distributive law $MH \xrightarrow{\lambda} HM$

Thank you!