From Coalgebraic to Monoidal Traces

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CMCS, Cyprus, 26/3/2010

Outline

Traced monoidal categories

Feedback operator in monoidal category:

$$\frac{X \otimes U \xrightarrow{\;f\;} Y \otimes U}{X \xrightarrow[\mathrm{Tr}(f)]{} Y}$$

Satisfying a bunch of requirements.

- Introduced by Joyal-Street-Verity, 1996
- Intuitive string notation possible

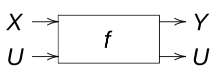
Informal distinction

- For "multiplicative" tensor $\otimes$
  wave-style trace
- For "additive" coproduct/biproduct $\oplus$,
  particle-style trace

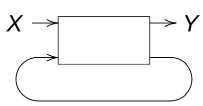
Here we use the latter, in Kleisli categories of monads.

Traces, pictorially

A map $X \otimes U \xrightarrow{f} Y \otimes U$ becomes:



And its trace $X \xrightarrow{\mathrm{Tr}(f)} Y$ is:



Standard examples

- Usual summation trace in finite-dimensional vector spaces:

$$\mathrm{Tr}\Big(X \otimes U \xrightarrow{M} Y \otimes U\Big)_{x,y} \;=\; \sum_u M_{x\otimes u, y\otimes u}$$

- Fixed point in Cpos with bottom:

$$\mathrm{Tr}\Big(X \times U \xrightarrow{f} Y \times U\Big)(x) \;=\; \pi_1\Big(\mathrm{fix}(\lambda(y,u).\, f(x,u))\Big)$$

- Iteration in pointed sets / sets with partial functions:

$$\mathrm{Tr}\Big(X + U \xrightarrow{f} Y + U\Big)(x) \;=\; \begin{cases} y & \text{if } \exists u_1,\ldots,u_n \in U.\, f(x) = u_1 \\ & \quad \ldots f(u_i) = u_{i+1} \ldots f(u_n) = y \\ \bot & \text{otherwise} \end{cases}$$

The latter will be generalised to Kleisli categories.

Basic construction

JSV'96: for a traced monoidal category $\mathbb{C}$, construct compact closed category $\mathrm{Int}(\mathbb{C})$ with:

Obj $(X^+, X^-) \in \mathbb{C} \times \mathbb{C}$

Mor $(X^+, X^-) \xrightarrow{f} (Y^+, Y^-)$ are $X^+ \otimes Y^- \xrightarrow{f} Y^+ \otimes X^-$ in $\mathbb{C}$.

- Composition via trace / feedback
- Applications in linear logic: game semantics / geometry of interaction.

Basic result [HJS'07]

Assuming:

- a monad $T: \mathbb{C} \to \mathbb{C}$ whose Kleisli category $\mathcal{K}\ell(T)$ is **CPO**$_\perp$-enriched
- a functor $F: \mathbb{C} \to \mathbb{C}$ with a distributive law $FT \Rightarrow TF$, giving a lifting $\overline{F}: \mathcal{K}\ell(T) \to \mathcal{K}\ell(T)$
- some minor details . . .

Initial algebra $F(A) \xrightarrow{\cong} A$ in $\mathbb{C}$ forms final coalgebra in $\mathcal{K}\ell(T)$

Concretely:

$$\frac{X \xrightarrow{c} T(F(X))}{X \xrightarrow{\mathrm{tr}(c)} T(A)}$$

Question

- Is there a connection between coalgebraic and monoidal traces?
- Or is this just a coincidence of names?

Iteration via coalgebraic traces I

- Assume $\mathbb{C}$ has (countable) coproducts $\coprod$
- The functor $Y + (-): \mathbb{C} \to \mathbb{C}$ then has an initial algebra:

$$\text{copower:} \quad \mathbb{N} \cdot Y \;=\; \textstyle\coprod_{i \in \mathbb{N}} Y$$

- There is always a distributive law $Y + T(X) \to T(Y + X)$.
- Hence we can apply the Trace Theorem: $\mathbb{N} \cdot Y$ is final in $\mathcal{K}\ell(T)$.

Iteration via coalgebraic traces II

$$\frac{X \xrightarrow{c} T(Y + X)}{X \xrightarrow{\mathrm{tr}(c)} T(\mathbb{N} \cdot Y)}$$

- Intuition:

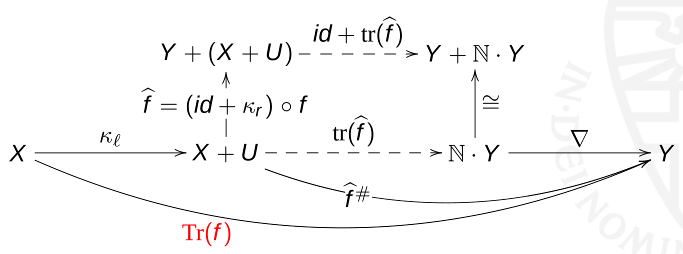
$$\mathrm{tr}(c)(x) = (n, y) \iff \begin{cases} y \text{ is reached after } n \text{ iterations} \\ \text{of } c \text{, starting in } x \end{cases}$$

- Post-composition with codiagonal $\mathbb{N} \cdot Y \xrightarrow{\nabla} Y$ ignores number of iterations, and yields an iteration operation:

$$\mathcal{K}\ell(T)(X, Y + X) \xrightarrow{(-)^{\#}} \mathcal{K}\ell(T)(X, Y)$$

Towards a monoidal trace

For $f: X + U \to Y + U$ in $\mathcal{K}\ell(T)$, by finality:



How this work progressed . . .

1. Coalgebraic trace definition $\mathrm{Tr}(f)$ from previous slide
2. Proof of trace properties: steady progress . . .
3. . . . failure with "vanishing II": trace over tensor is double trace
4. closer inspection showed:
   - special property of coproducts in $\mathcal{K}\ell(T)$ was needed
   - PhD thesis (Ottawa, 2000) of Esfandiar Haghverdi covers this: *every partially additive category is traced monoidal*
5. Hence: partial additivity in $\mathcal{K}\ell(T)$ is needed (and trace is derived from this partial additivity)

Initial & final objects

Assume $\mathbb{C}$ has finite products $(1, \times)$ and coproducts $(0, +)$.

LEMMA For a monad $T : \mathbb{C} \to \mathbb{C}$ the following are equivalent.

1. $T(0)$ is final, i.e. $T(0) \cong 1$
2. $0 \in \mathcal{K}\ell(T)$ is final (and hence zero-object)
3. $1 \in \mathrm{Alg}(T)$ is initial (and hence zero-object)

Additivity I

Assume $T(0)$ is final, and form "zero-map" in $\mathcal{K}\ell(T)$,

$$0 = \Big(X \xrightarrow{\;!\;} T(0) \xrightarrow{T(!)} T(Y)\Big)$$

Form "projections" for $+$ in $\mathcal{K}\ell(T)$,

$$p_1 = \Big(X + Y \xrightarrow{[\eta, 0]} T(X)\Big) \qquad p_2 = \Big(X + Y \xrightarrow{[0, \eta]} T(Y)\Big)$$

and then a special map, connecting coproducts and products:

$$\mathrm{bc} = \Big(T(X + Y) \xrightarrow{\langle \mu \circ T(p_1), \mu \circ T(p_2)\rangle} T(X) \times T(Y)\Big)$$

Additivity II

THEOREM [Coumans & Jacobs 2010]
The following are equivalent

1. These bc: $T(X + Y) \to T(X) \times T(Y)$ are isomorphism
2. $+$ in $\mathcal{K}\ell(T)$ is product (and hence biproduct)
3. $\times$ in $\mathrm{Alg}(T)$ is coproduct (and hence biproduct)

In this case we call the monad T additive.

EXAMPLES

- T is powerset: explains why categories of relations (Kleisli) and complete lattices (algebras) have biproducts
- T is multiset monad: explains why commutative monoids/groups and modules/vector spaces have biproducts.

Partial additivity

In this CMCS paper: call monad T partially additive if bc is cartesian natural transformation:

- each map bc: $T(X + Y) \to T(X) \times T(Y)$ is mono
- naturality squares are pullbacks

EXAMPLES Powerset, lift, distribution

For $f_i : X \to T(Y)$ say $\amalg_{i \in I} f_i = \nabla \circ b : X \to T(Y)$ exists if there is a "bound" $b : X \to T(I \cdot Y)$ exists with $p_i \circ b = f_i$.

This makes the Kleisli category partially additive.

Main result

Let T be a semi-additive monad whose Kleisli category $\mathcal{K}\ell(T)$ is $\mathbf{CPO}_\perp$ enriched, on a category $\mathbb{C}$ with (countable) coproducts, then:

- $\mathcal{K}\ell(T)$ is traced monoidal (via $+$, particle style)
- the monoidal trace $\mathrm{Tr}(f)$ can be described via the coalgebraic trace.

Applications

- Write $\mathcal{B}d(T) = \mathrm{Int}(\mathcal{K}\ell(T))$ for the category of "bidirectional monadic computation", with maps $X^{+} + Y^{-} \longrightarrow T(Y^{+} + X^{-})$

  For lift monad: Abramsky's category of games.

  Other examples, e.g. probabilistic games for distribution monad, require more study.

- Trace/feedback operator for coalgebraic components: new paper in preparation

Concluding remarks

- Coincidence of names not a coincidence!
- Basic properties of monads identified: (partial) additivity
- New class of examples of traced monoidal categories
- Follow-up work: bidirectional computation & traces for components.