Coalgebras and Modal Logics: an Overview

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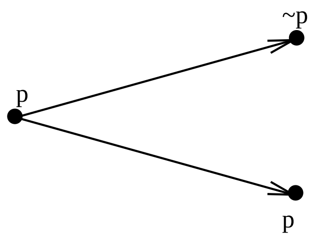
Part I: Examples

or:

Why should I care?

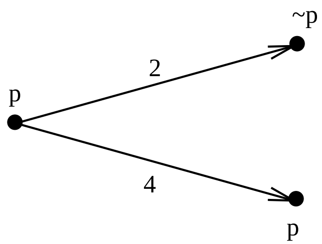
A Cook's Tour Through Modal Semantics

Kripke Frames.



$$C \to \mathcal{P}(C)$$

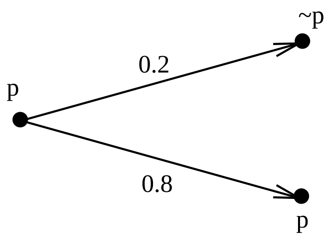
Multigraph Frames.



$$C \to \mathcal{B}(C)$$

$$\mathcal{B}(X) = \{f : X \to \mathbb{N} \mid \mathrm{supp}(f) \text{ finite}\}$$

Probabilistic Frames.



$$C \to \mathcal{D}(C)$$

$$\mathcal{D}(X) = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1\}$$

More Examples

Neighbourhood Frames.

$$C \to \mathcal{P}\mathcal{P}(C) = \mathcal{N}(C)$$

mapping each world $c \in C$ to a set of neighbourhoods

Game Frames over a set N of agents

$$C \to \{((S_n)_{n\in N}, f) \mid f : \prod_n S_n \to C\} = \mathcal{G}(C)$$

associating to each state $c \in C$ a *strategic game* with strategy sets S_n and outcome function f

Conditional Frames.

$$C \to \{f : \mathcal{P}(C) \to \mathcal{P}(C) \mid f \text{ a function}\} = \mathcal{C}(C)$$

where every state yields a *selection function* that assigns properties to conditions

Coalgebras and Modalites: A Non-Definition

Coalgebras are about *successors*. T-coalgebras are pairs (C, γ) where

$$\gamma : C \to TC$$

maps states to successors. Write $\mathsf{Coalg}(T)$ for the collection of T-coalgebras.

states $=$ elements $c \in C$	properties of states $=$ subsets $A \subseteq C$
successors $=$ elements $\gamma(c) \in TC$	properties of successors $=$ subsets $\heartsuit A \subseteq TC$

Modal Operators are about *properties* of successors, so

$$\frac{[\![\phi_1]\!], \dots, [\![\phi_n]\!] \subseteq C}{[\![\heartsuit(\phi_1, \dots, \phi_n)]\!] \subseteq TC}$$

with the intended interpretation $c \models \heartsuit(\phi_1, \dots, \phi_n)$ iff $\gamma(c) \in [\![\heartsuit\phi_1, \dots, \phi_n]\!]$.

Part II: Approaches to Syntax and Semantics

or:

What’s a modal operator?

Moss' Coalgebraic Logic: The Synthetic Approach

Idea. $\heartsuit$ reflects the action of T on sets: 'import' semantics into syntax

Concrete Syntax

$$\frac{\Phi \subseteq_f L}{\bigwedge \Phi \in L} \quad \frac{\phi \in L}{\neg\phi \in L} \quad \frac{\Phi \in T_\omega L}{\nabla\Phi \in L}$$

Abstract Syntax:

$$L \cong F(L) = \mathcal{P}_f(L) + L + T_\omega(L)$$

Modal Semantics

$$c \models \nabla\Phi \iff (\gamma(c), \Phi) \in T(\models)$$

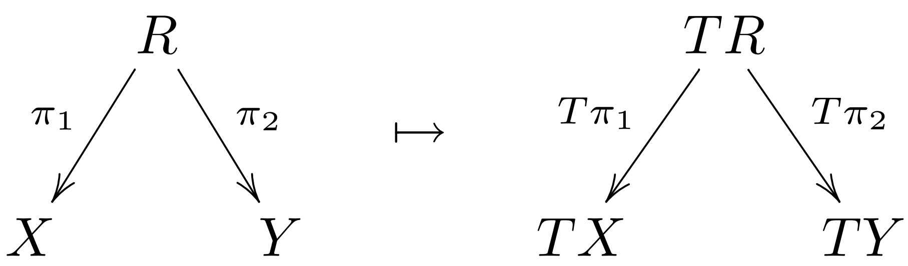
Algebraic Semantics

$$\begin{array}{ccc} F(L) & \longrightarrow & F(\mathcal{P}(C)) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \hat{\gamma}} \\ L & \xrightarrow[\llbracket\cdot\rrbracket]{} & \mathcal{P}(C) \end{array}$$

relative to T-coalgebra $(C, \gamma : C \to TC)$ where T_ω is the finitary part of T

Synthetic Semantics Explained

Relation Lifting: from *states* to *successors*



Formal Definition. (Assume T preserves *weak pullbacks* to make things work)

$$\overline{T}R = \{(T\pi_1(w), T\pi_2(w)) \mid w \in TR\} \subseteq TX \times TY$$

Modal Semantics. Assume that $\models$ is already given for 'ingredients' of $\alpha \in TL$

$$c \models \nabla\alpha \iff (\gamma(c), \alpha) \in \overline{T}(\models)$$

for $c \in C$ and $(C, \gamma : C \to TC) \in \mathsf{Coalg}(T)$.

Thm. [Moss, 1999] L has the Hennessy-Milner Property.

Example: Coalgebraic Logic of Multigraphs

Modal Operators for $\mathcal{B}X = \{f : X \to \mathbb{N} \mid \mathrm{supp}(f) \text{ finite}\}$

$$\frac{\alpha : L \to \mathbb{N} \text{ and } \mathrm{supp}(\alpha) \text{ finite}}{\nabla\alpha \in L}$$

Satisfaction. $c \models \nabla\alpha \iff (\gamma(c), \alpha) \in T(\models) \iff$ the 'magic square'

	x_1	x_2	$\cdots$	x_k	$\sum$
ϕ_1					w_1
$\vdots$					$\vdots$
ϕ_n					w_n
Σ	m_1	m_2	$\ldots$	m_n	

- $m_j = \gamma(c)(x_j)$ is multiplicity of x_j
- $w_i = \alpha(\phi_i)$ is weight of ϕ_i
- x/ϕ-entry is 0 if $x \not\models \phi$

can be filled according to the rules on the right.

Synthetic Semantics, Algebraically

Syntax as initial algebra. $L \cong \mathcal{P}_f(L) + LT(L)$

Semantics as algebra morphism

$$\begin{array}{ccc}
\mathcal{P}_f(L) + L + TL & \longrightarrow & \mathcal{P}_f(\mathcal{P}(C)) + \mathcal{P}(C) + T\mathcal{P}(C) \\
 & & \downarrow {\scriptstyle 1+1+\rho_C} \\
\downarrow {\scriptstyle i} & & \mathcal{P}_f(L) + \mathcal{P}(C) + \mathcal{P}(TC) \\
 & & \downarrow {\scriptstyle [\bigcap,(\cdot)^c,\gamma^{-1}]} \\
L & \xrightarrow[\llbracket\cdot\rrbracket]{} & \mathcal{P}(C)
\end{array}$$

where $\rho_C : T\mathcal{P}(C) \to \mathcal{P}(TC)$ is ' *lifted membership*', i.e.

$$\rho_C(\Phi) = \{t \in TC \mid (t, \Phi) \in \overline{T}(\in)\}$$

where $\epsilon_C \subseteq C \times \mathcal{P}(C)$ is membership (for $T = \mathcal{B}$ a 'magic square' problem)

Logics via Liftings: The Organic Approach

Idea. take $\heartsuit$ what we want it to mean: grow your own modalities

T-Structures then define the semantics of modalities: they

assign a *nbhd frame translation*

$$[\![\heartsuit]\!] : TC \to \mathcal{P}(\mathcal{P}(C)^n)$$

or, equivalently, a *predicate lifting*

$$[\![\heartsuit]\!] : \mathcal{P}(C)^n \to \mathcal{P}(TC)$$

to every modal operator $\heartsuit$ of the language, parametric in C.

Together with a T-coalgebra (C, γ) this gives (in the unary case) a

neighbourhood frame

$$C \xrightarrow{\gamma} TC \xrightarrow{[\![\heartsuit]\!]} \mathcal{P}\mathcal{P}(C)$$

boolean algebra with operator

$$\mathcal{P}(C) \xrightarrow{[\![\heartsuit]\!]} \mathcal{P}(TC) \xrightarrow{\gamma^{-1}} \mathcal{P}(C)$$

Induced **Coalgebraic Semantics** $[\![\phi]\!] \subseteq C$ of a modal formula

from a *modal perspective*	equivalent *algebraic viewpoint*
$c \in [\![\heartsuit\phi]\!]$ iff $[\![\phi]\!] \in [\![\heartsuit]\!] \circ \gamma([\![\phi]\!])$	$c \in [\![\heartsuit\phi]\!] \iff \gamma(c) \in [\![\heartsuit]\!]([\![\phi]\!])$

Example: The Logic of Multigraphs

Modal Operators for $\mathcal{B}X = \{\mu : X \to \mathbb{N} \mid \mathrm{supp}(\mu) \text{ finite}\}$

Our Choice. $\heartsuit(\phi, \psi)$, intended meaning 'at least 5 times as much ϕ's than ψ's'

Associated Lifting.

$$[\![\heartsuit]\!]_X(A, B) = \{\mu \in \mathcal{B}X \mid \mu(A) \geq 5 \cdot \mu(B)\}$$

where $\mu(A) = \sum_{x \in A} \mu(x)$

Satisfaction.

$$c \models \heartsuit(\phi, \psi) \iff \mu([\![\phi]\!]) \geq 5 \cdot \mu([\![\psi]\!])$$

where $\mu = \gamma(c)$ is the local weighting as seen from point c.

(i.e. one *can* pick and choose the primitives but *has to* define their meaning)

Part III: Reasoning in Coalgebraic Logics

or:

What's a good proof system?

Synthetic Approach: One Proof Calculus for All

Recall. Semantics as algebra morphism

$$
\begin{array}{ccc}
\mathcal{P}_f(L) + L + TL & \longrightarrow & \mathcal{P}_f\mathcal{P}(C) + \mathcal{P}(C) + T\mathcal{P}(C) \\
 & & \downarrow {\scriptstyle 1+1+\rho_C} \\
\downarrow i & & \mathcal{P}_f\mathcal{P}(C) + \mathcal{P}(C) + \mathcal{P}T(C) \\
 & & \downarrow {\scriptstyle [\bigcap, (\cdot)^c, \gamma^{-1}]} \\
L & \xrightarrow[\llbracket \cdot \rrbracket]{} & \mathcal{P}(C)
\end{array}
$$

where $\rho_C : T\mathcal{P}(C) \to \mathcal{P}(TC)$ is $\rho_C(\Phi) = \{t \in TC \mid (t, \Phi) \in \overline{T}(\in)\}$

Slim Redistributions. 'import' the action of ρ into the proof system.

$$\Phi \in T\mathcal{P}(X) \text{ redistribution of } A \in \mathcal{P}(TX) \iff A \subseteq \rho_X(\Phi)$$

Call Φ *slim* if $\Phi \in \mathcal{P}_\omega T_\omega(A)$ (i.e. Φ only re-arranges material from A)

Notation. $\mathsf{SRD}(A) = \{\Phi \in T\mathcal{P}(A) \mid \Phi \text{ slim redistribution of } A\}$

Redistributions of Multisets

Redistributions of $\mathcal{B}X = \{f : X \to \mathbb{N} \mid \mathrm{supp}(f) \text{ finite}\}$

$\Phi : \mathcal{P}(X) \to_f \mathbb{N} \in \mathcal{BP}X$ *redistribution* of $A \in \mathcal{P}(X \to_f \mathbb{N}) = \mathcal{P}(\mathcal{B}X)$

$$\Longleftrightarrow$$

A only contains $f : X \to_f \mathbb{N}$ that allow to fill the 'magic square'

	x_1	x_2	$\cdots$	x_k	$\sum$
S_1					w_1
$\vdots$					$\vdots$
S_n					w_n
Σ	m_1	m_2	$\ldots$	m_n	

- x/S-entry is 0 if $x \notin S$
- m_j is f-multiplicity of x_j
- w_i is Φ-weight of S_i

Φ is *slim* if each nozero S_i only contains nonzero x_js relative to some element of A

The Synthetic Proof System

Synthetic Proofs.

- judegements are inequalities $a \leq b$ for $a, b \in L$
- propositional logic and cut: from $a \leq b$ and $b \leq c$ infer $a \leq c$

Modal Proof Rules.

$$(\nabla 1) \quad \frac{\alpha \overline{\leq} \beta}{\nabla \alpha \leq \nabla \beta} \qquad (\nabla 4) \quad \frac{\{a \wedge \nabla \alpha' \leq \bot \mid \alpha' \in T_\omega(\phi) \setminus \{\alpha\}\} \qquad \top \leq \bigvee \phi}{a \leq \nabla \alpha}$$

$$(\nabla 2) \quad \frac{\{\nabla (T \bigwedge)(\Phi) \leq a \mid \Phi \in \mathsf{SRD}(A)\}}{\bigwedge \{\nabla \alpha \mid \alpha \in A\} \leq a} \qquad (\nabla 3) \quad \frac{\{\nabla \alpha \leq a \mid (\alpha, \Phi) \in \overline{T}(\in)\}}{\nabla (T \bigvee) \Phi \leq a}$$

where $a \in L, \alpha, \beta \in T_\omega L, A \in \mathcal{P}_\omega T_\omega(L)$ and $\Phi \in T_\omega \mathcal{P}_\omega(L)$.

Thm. [Kupke, Kurz, Venema 2009] The synthetic system is sound and complete over T-coalgebras.

Organic: Proof Systems for Homegrown Modalities

Recall. Language L given by operators $\heartsuit$, semantics by $[\![\heartsuit]\!] : \mathcal{P}(X) \to \mathcal{P}(TX)$

Proof Systems in terms of sequents: $\Gamma \subseteq L$ with $[\![\Gamma]\!] = \bigcup\{[\![A]\!] \mid A \in \Gamma\}$

One-step Rules (*specific* for each choice of $\heartsuit$s)

$$\frac{\Gamma_1 \quad \dots \quad \Gamma_n}{\Gamma_0} \quad \sim \quad \frac{\text{property of } \textit{states}}{\text{property of successors}} \quad \sim \quad \frac{[\![\Gamma_1]\!] \cap \dots \cap [\![\Gamma_n]\!] \subseteq X}{[\![\Gamma_0]\!] \subseteq TX}$$

where

- $\Gamma_1, \dots, \Gamma_n \subseteq V \cup \neg V$ are propositional over a set V of variables
- $\Gamma_0 \subseteq \{\heartsuit(p_1, \dots, p_n) \mid \heartsuit\ n\text{-ary}\} \cup \{\neg\heartsuit(p_1, \dots, p_n) \mid \heartsuit\ n\text{-ary}\}$

Crucial: need *Coherence Conditions* between proof rules and semantics

Organic Modalities: Coherence Conditions

Consider a set X and a valuation $\tau : V \to \mathcal{P}(X)$.

Coherence: matching between rules and semantics at *one-step level*

Propositional Sequents $\Gamma \subseteq V \cup \neg V$

$$\Gamma \ \tau\text{-valid} \iff [\![\Gamma]\!]_\tau = X \text{ where } [\![p]\!]_\tau = \tau(p)$$

Modalised Sequents $\Gamma \subseteq \{\pm\heartsuit(p_1, \ldots, p_n) \mid \heartsuit \ n\text{-ary}\}$

$$\Gamma \ \tau\text{-valid} \iff [\![\Gamma]\!]_\tau = TX \text{ where } [\![\heartsuit(p_1, \ldots, p_n)]\!]_\tau = [\![\heartsuit]\!](\tau(p_1), \ldots, \tau(p_n))$$

where $\pm$ indicates possible negation.

Coherence relates τ-validity of premises with τ-validity of conclusions

Organic Modalities: Coherence Conditions

One-Step Soundness of a set $\mathcal{R}$ of one-step rules: for all $\tau : V \to \mathcal{P}(X)$

$$\Gamma_1, \dots, \Gamma_n \ \tau\text{-valid} \implies \Gamma_0 \ \tau\text{-valid}$$

for all $\Gamma_1 \dots \Gamma_n / \Gamma_0 \in \mathcal{R}$

One-Step Completeness of a set $\mathcal{R}$ of one-step rules: for all $\tau : V \to \mathcal{P}(X)$

$$\Gamma \ \tau\text{-valid} \implies \exists \frac{\Gamma_1 \dots \Gamma_n}{\Gamma_0} \in \mathcal{R} \ (\Gamma_i \sigma \ \tau\text{-valid and } \Gamma_0 \sigma \subseteq \Gamma)$$

for some renaming $\sigma : V \to V$, for all $\Gamma \subseteq_f \{\pm \heartsuit(p_1, \dots, p_n) \mid \heartsuit \ n\text{-ary}\}$.

Thm. [P, 2003, Schröder 2007] One-step soundness and one-step completeness imply soundness and (cut-free) completeness, respectively, when augmented with propositional reasoning.

Organic Logics for Multisets

Proof Rules for $\mathcal{B}X = \{\mu : X \to \mathbb{N} \mid \mathrm{supp}(f) \text{ finite}\}$

Modal Operators

$$\Lambda = \{L_p(c_1, \ldots, c_m) \mid n \in \mathbb{N}, p_1, \ldots, p_m \in \mathbb{Z}\}$$

Intended Meaning.

$$[\![L_p(c_1, \ldots, c_m)]\!](S_1, \ldots, S_m) = \{\mu \in \mathcal{B}X \mid \sum_{j=1}^{m} c_j \cdot \mu(S_j) \geq p\}$$

Sound and Complete Proof Rules. (subject to arith. side condition)

$$\frac{\sum_{i=1}^{n} r_i \cdot \sum_{j=1}^{m_i} c_i^j a_i^j \geq 0}{\{\mathsf{sg}(r_i) L_{p_i}(c_1^i, \ldots, c_{m_i}^i)(a_i^1, \ldots, a_i^{m_i}) \mid i = 1, \ldots, n\}}$$

- $\mathsf{sg}(r)A = A$ if $r > 0$ and $\mathsf{sg}(r)A = \neg A$ if $r < 0$
- premise reflects arithmetic of characteristic functions as propositional formula

Part IV: Automated Reasoning in Coalgebraic Logics

or:

How do I mechanise satisfiability?

Synthetic: Automata for Modal Formulas

Idea. Formulas $\phi \leftrightarrow$ Automata $\mathbb{A}_\phi$ so that

$$(c, C) \models \phi \iff \mathbb{A} \text{ accepts } (c, C)$$

where $C = (C, \gamma)$ is a T-coalgebra and $c \in C$.

Satisfiability checking via automata: ϕ satisfiable $\iff L(\mathbb{A}_\phi) \neq \emptyset$

Coalgebra Automata are tuples $\mathbb{A} = (A, a_i, \Delta, \Omega)$ where

- A is a finite set of states and $a_I \in A$ is initial
- $\Delta : A \to \mathcal{P}\mathcal{P}(TA)$ is the transition function
- $\Omega : A \to \mathbb{N}$ is a parity function

(we think of these automata as *alternating* due to layering of $\mathcal{P}$)

Acceptance via Parity Games

Given. $\mathbb{A} = (A, a_i, \Delta, \Omega)$ and state c of T-coalgebra (C, γ).

Acceptance. $\mathbb{A}$ accepts c if $\exists$ has a winning strategy from (a_I, c) on the board

$$B = (A \times C) \cup (TA \times TC) \cup (\mathcal{P}TA \times C) \cup \mathcal{P}(A \times C$$

where legal moves are as follows:

Position	Player	Moves	Priority
$(a, c) \in A \times C$	$\exists$	$\{(\Xi, c) \in \mathcal{P}(TA) \times C \mid \Xi \in \Delta(a)\}$	$\Omega(a)$
$(\Xi, c) \in \mathcal{P}T(A) \times C$	$\forall$	$\{(\xi, \tau) \in TA \times TC \mid \xi \in \Xi, \tau = \gamma(c)\}$	0
$(\xi, \tau) \in TA \times TC$	$\exists$	$\{Z \in \mathcal{P}(A \times C) \mid (\xi, \tau) \in \overline{T}Z\}$	0
$Z \in \mathcal{P}(A \times C)$	$\forall$	Z	0

Intuition. (Recall $\Delta : A \to \mathcal{P}\mathcal{P}TA$)

- $\Delta(a) \sim$ formula in DNF: $\exists$ chooses disjunct, $\forall$ chooses element
- 'modal' steps lift acceptance relation and attract priorities

Automata and Fixpoint Logic

Modal Language. Positive Logic + ∇ + *fixpoint formulas*

$$\mu L ::= x \mid \top \mid \bot \mid \phi \wedge \psi \mid \phi \vee \psi \mid \nabla\alpha \mid \mu x.\phi \mid \nu x.\phi$$

where $\alpha \in T_\omega L$ and $x \in V$ is a variable.

Semantics. As before, with μ/ν interpreted as least/greatest fixpoints.

Thm. [Venema, 2008] For every $\phi \in \mu L$ there exists $\mathbb{A}_\phi$ such that

$$\mathbb{A}_\phi \text{ accepts } (c, C) \iff c \models \phi$$

and vice versa. That is: *Automata are Formulas are Automata*.

Intuition.

- loops in the automaton $\sim$ unfolding of fixpoints
- parity condition: only *finite* unfoldings of *least* fixpoints

Organic: Tableau Calculi

Here. Easier to use *Tableaux* than *Sequent Calculi*

Formulas.

$$L \ni A, B ::= p \mid \overline{p} \mid A \wedge B \mid A \vee B \mid \heartsuit(A_1, \dots, A_n) \mid \eta p.A$$

where $\heartsuit$ is n-ary and $\eta \in \{\mu, \nu\}$

Tableau Sequents. Finite sets of formulas $\Gamma = \{A_1, \dots, A_n\}$ read *conjunctively*

Tableau Rules. As before, with modal rules dualised

$$\frac{\Gamma; A \wedge B}{\Gamma; A; B} \quad \frac{\Gamma; A \vee B}{\Gamma; A \quad \Gamma; B} \quad \frac{\Gamma; \eta p.A}{\Gamma; A[p := \eta p.A]} \quad \frac{\Gamma_0\sigma, \Delta}{\Gamma_1\sigma \dots \Gamma_n\sigma} \quad \frac{\Gamma, A, \overline{A}}{}$$

Remarks.

- Expansion only ever creates *finitely* many formulas
- *No distinction* between *least* and *greatest* fixpoints

Satisfiability via Games

As before. Two-Player Parity Games

- every board position b has a priority $\Omega(b)$
- $\exists$ wins (and $\forall$ looses) a play if largest infinitely occurring priority is even
- unfolding of *least fixpoints* gives *odd* priorities

Model Checking Game

- modal satisfiability game
- played on state/formula pairs
- unfolding of fixpoints

Tableaux Game

- played on sequents and rules
- $\forall$ chooses rule
- $\exists$ chooses conclusion

Thm. [Cîrstea, Kupke, P 2009] A formula is satisfiable if it has a closed tableau.

Part V: Other Aspects of Coalgebraic Logics

or:

What is there that I didn’t comment on?

Other Aspects

Coalgebraic Logics, Categorically.

- Logics via Adjunctions
  [Klin, Kurz, Jacobs, Sokolova]
- Logics via Presentations
  [Bonsangue, Kurz]

Compositionality

- Logics for Composite Functors
  [Cîrstea, P, Schröder]

Proof Theory.

- Sequents for ∇
  [Bílková, Palmigiano, Venema]
- Interpolation [P, Schröder]

Synthetic vs Organic.

- back and forth [Leal]

Complexity.

- via Tableaux
  [Cîrstea, Kupke, Schröder, P]

Extensions of Set-based logics.

- Hybridisation
  [Myers,Kupke,P,Schröder]
- Global Consequence
  [Goré,Kupke,P]
- Path-Based Logics [Cîrstea]

Part VI: Perspectives

or:

What should we think about in the future?

Some Biased Food for Thought

Coalgebraic Logics are Feature-Rich, Compositional and Decdiable

Strategic.

- *Implement:* Demonstrate techniques on non-trivial problems
- *Apply:* Use coalgebraic logics in modelling and verification

Technical.

- *Understand:* relationship between Tableaux and Automata
- *Deepen:* (Automated) reasoning with frame conditions

Conceptual.

- *Generalise:* How about e.g. MV-algebras modelling uncertainty?
- *Learn:* Adapt ILP Techniques to enable machine learning

Last Part: Questions

and:

Thanks for your attention!