# Recursive Program Schemes and Context-Free Monads

Jiří Adámek

Stefan Milius

Jiří Velebil

mail@stefan-milius.eu

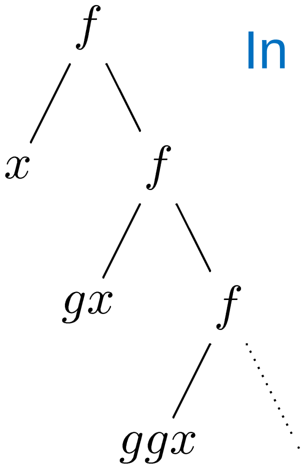
# What is a context-free tree?

Algebraic Trees of B. Courcelle = all solutions of recursive program schemes

**Example.**

$$\varphi(x) = f(x, g(\varphi(x))) \quad \rightarrow$$

variable givens



**In general.**

$\Phi$ signature of variables;
$\Sigma$ signature of givens

$$\varphi_i(\vec{x}) = t_i(\vec{x}), \qquad i = 1, \dots, n$$

**Guardedness:**

$t_i$'s head symbol is from $\Sigma$.

**Theorem.** Every guarded recursive program scheme has a unique solution.

Context-Free (= algebraic) trees: 1) form an iterative algebraic theory.
2) are closed under 2nd-order substitution.

**This talk:** Construction of a „context-free“ monad for every endofunctor.

# Background & History

- Infinite Trees & Algebraic Semantics: B. Courcelle (1983), I. Guessarian (1981)
    - General Algebra: Signatures, Sets, Trees
- E. Badouel (1989): Infinite Trees form a monad.

Coalgebraic Approach:

- L. Moss: Parametric Corecursion, TCS 2001.
- N. Ghani et al (CMCS 2001), P. Aczel + AV (CMCS 2001)
- P. Aczel + AMV: Infinite Trees a Coalgebraic View (TCS 2003)
- AMV: Iterative Algebras & Rational Trees (CMCS‘04)
- Ghani et al: Solving Algebraic Equations using Coalgebra (TIAA 2003)
- L. Moss & M: The category theoretic solution of recursive program schemes

Here: apply ideas from these.

# Outline

- Infinite Trees Coalgebraically
- Rational Trees Coalgebraically
- Context-Free Trees Coalgebraically

# Infinite Trees Coalgebraically

Set → category $\mathcal{A}$ with $+$

$\Sigma$ signature → $H : \mathcal{A} \to \mathcal{A}$

e. g. $H_\Sigma X = \coprod_{n \in \mathbb{N}} \Sigma_n \times X^n$

$T_\Sigma X =$ all $\Sigma$-trees on $X$ → final coalgebra for $H(-) + X$

**Assumption.** $\forall X \in |\mathcal{A}| \; \exists \, TX$ final coalgebra for $H(-) + X$

**Theorem.** 1. $TX \cong HTX + X$ (Lambek's Lemma)

variables and non-variables separated nicely

2. $T \cong HT + Id$ is an ideal monad.

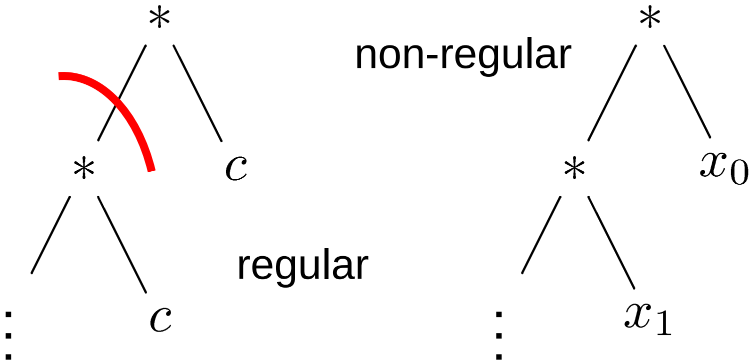
3. $T$ is the free completely iterative monad on $H$

unique solutions of 1st-order recursive eqns

# Rational Trees Coalgebraically

**Goal**: Abstract description of regular trees.

E. g. $\Sigma = \{\, *, c \,\}$



**Assumption.** $\mathcal{A}$ locally finitely presentable category, $H : \mathcal{A} \to \mathcal{A}$ finitary

**Construction.**

$$\mathrm{EQ}_A : \quad \begin{tikzcd} X \arrow[r, "e"] \arrow[d, "h"'] & HX + A \arrow[d, "Hh+A"] \\ Y \arrow[r, "f"'] & HY + A \end{tikzcd} \quad \xmapsto{\mathrm{Eq}_A} \quad \begin{tikzcd} X \arrow[d, "h"] \\ Y \end{tikzcd}$$

finitely presentable

$$RA \overset{\text{def}}{=} \operatorname{colim} \left( \mathrm{EQ}_A \xrightarrow{\mathrm{Eq}_A} \mathcal{A} \right)$$

**Theorem.**

1. $RA \cong HRA + A$,
2. $R \cong HR + Id$ is an ideal monad,
3. $R$ is the free iterative monad on $H$.

# Alternative & Examples

**Equivalently:** $\mathcal{A} = \mathsf{Set}$

**Definition.** $X \xrightarrow{c} HX$ locally finite $:\iff$ $\forall x \in X. \langle x \rangle \subseteq X$ finite

**Theorem.** 1. $R\emptyset$ is the final locally finite coalgebra for $H$.

2. Similarly for arbitrary lfp categories $\mathcal{A}$.

**Examples.** $\mathcal{A} = \mathsf{Set}$

$HX = \{0,1\} \times X^A$ $\quad R\emptyset = $ regular languages

$HX = H_\Sigma X = \coprod_{n\in\mathbb{N}} \Sigma_n \times X^n$ $\quad RX = $ rational $\Sigma$-trees on $X$

$HX = \{\{x,y\} \mid x,y \in X\}$ $\quad RX = $ rat. unord. bin. trees on $X$

$HX = \mathbb{R} \times X$ $\quad R\emptyset = $ eventually periodic streams

$\mathcal{A} = \mathsf{Vec}_\mathbb{R}$ $\quad HX = \mathbb{R} \times X$ $\quad R0 = $ all rational streams

$\mathcal{A} = \mathsf{Set}^\mathcal{F}$ $\quad HX = X \times X + \delta X$ $\quad RV = $ rational $\lambda$-trees up to $\alpha$-eq.

$V : \mathcal{F} \to \mathsf{Set} \quad V(\Gamma) = \Gamma$

# Recursive Program Schemes

RPS‘s categorically:

natural in X

(Σ+Φ)-terms on X

$$\varphi_i(\vec{x}) = t_i(\vec{x}), \quad i = 1, \dots, n \quad \rightsquigarrow$$

$$\begin{tikzcd}
H_\Phi X \arrow[r, "e"] \arrow[dr, "\exists"'] & F_{\Sigma+\Phi} X \\
& H_\Sigma F_{\Sigma+\Phi} X + X \arrow[u, "\mathsf{can}"']
\end{tikzcd}$$

guardedness

**Assumptions.** $\mathcal{A}$ locally finitely presentable

coproduct injections are monos and monos closed under $+$

$V, H, \dots : \mathcal{A} \to \mathcal{A}$ finitary

$\rightsquigarrow F^{H+V}$ free monad $\qquad H \xrightarrow{\kappa} T$ free completely iterative monad on $H$

**Definition.** rps:

$$\begin{tikzcd}
V \arrow[r, "e"] \arrow[dr, "\exists"'] & F^{H+V} \\
& HF^{H+V} + Id \arrow[u, "\mathsf{can}"']
\end{tikzcd}$$

guarded rps

solution in $H \xrightarrow{\sigma} S$:

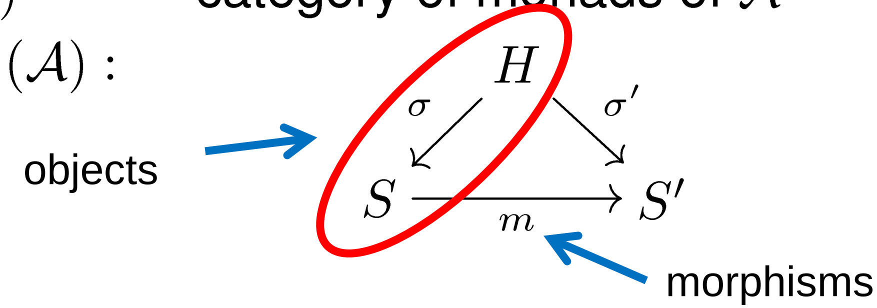
$$\begin{tikzcd}
V \arrow[r, "e^\dagger"] \arrow[d, "e"'] & S \\
F^{H+V} \arrow[ur, "\widehat{[\sigma, e^\dagger]}"'] &
\end{tikzcd} \qquad \begin{tikzcd}
S \\
H + V \arrow[u, "{[\sigma, e^\dagger]}"']
\end{tikzcd}$$

**Theorem.** Every guarded rps has a unique solution in $(T, \kappa)$. (based on Ghani et al)

# Preliminaries for context free monads

**Notation.** $\mathsf{Mon}(\mathcal{A}) \quad = \quad$ category of monads of $\mathcal{A}$

$H/\mathsf{Mon}(\mathcal{A})$ :



objects

morphisms

$$\begin{tikzcd}
 & H \arrow[dl, "\sigma"'] \arrow[dr, "\sigma'"] & \\
S \arrow[rr, "m"'] & & S'
\end{tikzcd}$$

$$\mathcal{H}(H \xrightarrow{\sigma} S) \quad = \quad H \xrightarrow{H\eta^S} HS \xrightarrow{\mathsf{inl}} HS + Id$$

**Theorem.** (Ghani et al)

1. $\mathcal{H}$ is an endofunctor of $H/\mathsf{Mon}(\mathcal{A})$.

2. $$\begin{tikzcd}
 & H \arrow[dl, "\kappa"'] \arrow[dr, "\mathsf{inl}\cdot H\eta^T"] & \\
T \arrow[rr, "\cong"'] & & HT + Id
\end{tikzcd}$$ is the final $\mathcal{H}$-coalgebra.

# Construction of the „context-free“ monad

1. for guarded rps

$$\begin{tikzcd} V \arrow[r, "e"] \arrow[dr, "e_0"'] & F^{H+V} \\ & HF^{H+V}+Id \arrow[u, "\text{can}"'] \end{tikzcd}$$

finitely presentable

$$\frac{V \xrightarrow{e_0} HF^{H+V}+Id}{F^{H+V} \xrightarrow{\bar{e}} HF^{H+V}+Id}$$

$\mathcal{H}$-coalgebra

2. $\mathrm{EQ}_0 \subseteq \mathrm{Coalg}(\mathcal{H})$ given by all coalgebras $F^{H+V} \xrightarrow{\bar{e}} HF^{H+V}+Id$

3. $\widetilde{C} \quad = \quad \mathrm{colim}\big(\mathrm{EQ}_0 \longrightarrow \mathrm{Mon}(\mathcal{A})\big)$

forgetful functor

4. $H \xrightarrow{\tilde{\varrho}} \widetilde{C}$ is an $\mathcal{H}$-coalgebra $\rightsquigarrow$ $\widetilde{C} \xrightarrow{\tilde{c}} T$ unique homomorphism

5. factorization in $\mathrm{Mon}(\mathcal{A})$:

$$\begin{tikzcd} \widetilde{C} \arrow[rr, "\tilde{c}"] \arrow[dr, two heads, "k"'] & & T \\ & C \arrow[ur, tail, "c"'] & \end{tikzcd}$$

strong epi

mono

Observation. C is countably accessible.

# Main results

$$\varrho = \left( H \xrightarrow{\tilde{\varrho}} \widetilde{C} \xrightarrow{k} C \right)$$

**Theorem.**

1. Every guarded recursive program scheme has a unique solution in $(C, \varrho)$.
2. $C \cong HC + Id$ is an ideal monad.
3. For $\mathcal{A} = \mathsf{Set}$, $H = H_\Sigma$ we have:

$$CX \quad = \quad \text{all context-free } \Sigma\text{-trees on } X.$$

# Summary

- Context-free trees are precisely the solutions of rps‘s.
- All infinite trees are captured by the free completely iterative monad T.
- All rational trees are captured by the rational monad R.
- Context-free trees are captured by the context-free monad C.

# Open problems

- Is C an iterative monad in the sense of Calvin Elgot?
- Closedness under 2nd-order substitution?
- Universal property of the context-free monad?
- Further examples.