Higher-order algebras and coalgebras from parameterized endofunctors

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- Higher-order & parameterized endofunctors
- Initial and final suitability

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Higher-order endofunctors

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- Endofunctor category, $\mathrm{End}(\mathcal{C}) = [\mathcal{C}, \mathcal{C}]$.
- Monad category, $\mathsf{Mon}(\mathcal{C})$ (abusing terminology slightly)

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- There are more constrained notion of parameterized endofunctors, (e.g. structural actions (Blute-Cockett-Seely '97), parameterized monads (Uustalu '03, Atkey '06), etc.)
- We use the unconstrained definition studied by Kurz and Pattinson '00.

Parameterized endofunctors to higher-order endofunctors

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For $F\colon \mathcal{B} \times \mathcal{C} \to \mathcal{C}$, let $\widehat{F}\colon [\mathcal{B}, \mathcal{C}] \to [\mathcal{B}, \mathcal{C}]$ be given by

$$\widehat{F}(X)(b) = F(b, Xb)$$

for $X\colon \mathcal{B} \to \mathcal{C}$ and $b \in \mathcal{B}$. $\widehat{F}$ is the *higher-order endofunctor generated by the parameterized endofunctor* F.

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Goal

Characterize initial algebras and final coalgebras of these higher-order endofunctors in terms of lower-order properties.

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- *finally suitable* if $F(b, -)$ admits a final coalgebra for any $b \in \mathcal{B}$.

Suitability to Higher-order Algebras and Coalgebras

An initially suitable parameterized endofunctor $F\colon \mathcal{B} \times \mathcal{C} \to \mathcal{C}$ induces a $\mathcal{C}$-endofunctor $\mathcal{R}_F$:

$$\begin{tikzcd}
F(x, \mathcal{R}_F x) \arrow[rr, "r_x"] \arrow[d, "F(x,\mathcal{R}_F f)"'] \arrow[dr, dotted, "F(f,\mathcal{R}_F f)=F(-,\mathcal{R}_F -)f"] & & \mathcal{R}_F x \arrow[d, "\mathcal{R}_F f"] \\
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- $\widehat{F}(R_F) = F(-,\mathcal{R}_F -) \overset{r}{\Longrightarrow} \mathcal{R}_F$ is a natural isomorphism.

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- $\widehat{F}\mathcal{R}_F \overset{r}{\Longrightarrow} \mathcal{R}_F$ is an $\widehat{F}$-algebra!

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- $\widehat{F}\mathcal{R}_F\overset{r}{\Longrightarrow}\mathcal{R}_F$ is an $\widehat{F}$-algebra!
- Let $\mathcal{S}_F\overset{s}{\Longrightarrow}\widehat{F}\mathcal{S}_F$ be the $\widehat{F}$-coalgebra induced by finally suitable parameterized endofunctors.

The Punchline

Theorem (J.Kim '09)

Let $\widehat{F}$ be a $[\mathcal{B},\mathcal{C}]$-endofunctor generated by a parameterized endofunctor $F\colon \mathcal{B}\times\mathcal{C}\to\mathcal{C}$. The following are equivalent:

1. *F is initially (resp. finally) suitable.*
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- If F is finally suitable, $\mathcal{S}_F \overset{s}{\Rightarrow} \widehat{F}\mathcal{S}_F$ is the final $\widehat{F}$-coalgebra.
- The result can be specialized to "parameterized monads" $F \colon \mathcal{B} \to \mathrm{Mon}(\mathcal{C})$. A monad structure can be imposed on the higher-order $[\mathcal{B}, \mathcal{C}]$-endofunctor $\widehat{F}$.

An Example of a Parameterized Endofunctor I

Example

Let $G_1 \overset{\theta}{\Longrightarrow} G_0$ be a natural transformation between two $\mathcal{C}$-endofunctors. Let $D\colon \mathbf{2} \times \mathcal{C} \to \mathcal{C}$ be given by

$$D(i, x) = G_i(x) \qquad \text{and} \qquad D(!, x) = \theta_x$$

for $i \in \mathbf{2}$, $x \in \mathcal{C}$. Recall $\mathbf{2} = \begin{tikzcd} 0 \arrow[loop below, "\mathrm{id}_0"] & 1 \arrow[l, "!"'] \arrow[loop below, "\mathrm{id}_1"] \end{tikzcd}$.

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- D is initially suitable if G_0 and G_1 admit initial algebras.
- D is finally suitable if G_0 and G_1 admit final coalgebras.
- The initiality version of the theorem generalizes a result by Chuang and Lin '06, proved for arrow categories.

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Example

Let H be a $\mathcal{C}$-endofunctor. Let $E\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be given by

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for $a, x \in \mathcal{C}$.

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- E is finally suitable $\Longleftrightarrow$ E is iteratable
- The finality version of the theorem generalizes a result by Aczel, Adámek, Milius, Velebil '03, proved for iteratable endofunctors.

An Example of a Parameterized Endofunctor III

Example

Let A, B be nonempty sets. Let $F\colon (\mathsf{Set}^{\mathsf{op}} \times \mathsf{Set}) \times \mathsf{Set} \longrightarrow \mathsf{Set}$ be given by $F(\langle A, B\rangle, C) = (B \times C)^A$.

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- F is finally suitable.
- $\mathcal{S}_F\langle A, B\rangle = \Gamma_{A,B} = \{f\colon A^\omega \to B^\omega : f \text{ causal}\}$. Let $\Gamma_{A,B} \xrightarrow{\gamma_{A,B}} (B \times \Gamma_{A,B})^A$ be given by

$$\gamma_{A,B}(f)(a) = \langle \mathsf{hd} \circ f \circ c_a, \mathsf{tl} \circ f \circ c_a\rangle$$

for $f \in \Gamma_{A,B}$, $a \in A$, $c_a(\sigma) = a{:}\sigma$. (Rutten)

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- Let $\mathrm{Hom} \overset{e}{\Rightarrow} \widehat{F}\,\mathrm{Hom} = F(-, \mathrm{Hom}\,-)$ be a $\widehat{F}$-coalgebra where the components

$$\mathrm{Hom}(A, B) \xrightarrow{e_{\langle A,B\rangle}} (B \times \mathrm{Hom}(A, B))^A$$

is given by

$$e_{\langle A,B\rangle}(f)(a) = \langle f(a), f\rangle.$$

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- $m_{A,B}(f)(\sigma_0, \sigma_1, \sigma_2, \ldots) = (f(\sigma_0), f(\sigma_1), f(\sigma_2), \ldots)$

map as a higher-order coalgebra morphism II

By finality of γ, there is a higher-order coalgebra morphism $\mathrm{Hom} \overset{m}{\Longrightarrow} \Gamma$:

- $$\begin{tikzcd} \mathrm{Hom}(A,B) \arrow[r, "e_{\langle A,B\rangle}"] \arrow[d, "m_{A,B}"'] & (B \times \mathrm{Hom}(A,B))^A \arrow[d, "(B\times m_{A,B})^A"] \\ \Gamma_{A,B} \arrow[r, "\gamma_{A,B}"'] & (B \times \Gamma_{A,B})^A \end{tikzcd}$$
- $m_{A,B}(f)(\sigma_0, \sigma_1, \sigma_2, \ldots) = (f(\sigma_0), f(\sigma_1), f(\sigma_2), \ldots)$
- $m_{A,B}$ is map!

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Conclusions

1. Algebraic and coalgebraic properties of higher-order endofunctors should be studied.
2. Particulars of other “constrained” functor categories should be studied.