Interpretations as coalgebra morphisms

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Outline

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Abstract definitions of logic

Abstract Logic as a consequence relation

$$\mathcal{A} = \langle A, \vdash_{\mathcal{A}} \rangle,$$

where $\vdash_A: \mathcal{P}(A) \times A$ is a consequence relation in A.

Abstract Logic as a closure operator

$$\mathcal{A} = \langle A, C_{\mathcal{A}} \rangle,$$

where $C_{\mathcal{A}}$ is a closure operator, i.e., a mapping $C_{\mathcal{A}} : \mathcal{P}(A) \to \mathcal{P}(A)$ such for that for all $X, Y \subseteq A$,

1. $X \subseteq C_{\mathcal{A}}(X)$;
2. $X \subseteq Y \Rightarrow C_{\mathcal{A}}(X) \subseteq C_{\mathcal{A}}(Y)$;
3. $C_{\mathcal{A}}(C_{\mathcal{A}}(X)) = C_{\mathcal{A}}(X)$.

Abstract definitions of logic

Abstract Logic as a closure system

$$\mathcal{A} = \langle A, \mathcal{T}_\mathcal{A} \rangle$$

where $\mathcal{T}_\mathcal{A}$ is a closure system on A, i.e., a family $\mathcal{F}$ of subsets of A closed under arbitrary intersections (here we consider $\bigcap \varnothing = A$).

Theorem

Let A be a set. For each closure operator $C_\mathcal{A}$ in A we can associate a closure system $\mathcal{T}_\mathcal{A}$ and, conversely, for each closure system $\mathcal{T}_\mathcal{A}$ a closure operator $C_\mathcal{A}$ in such way that they are mutually inverses of one another:

$$\begin{array}{lcl} C_\mathcal{A} & \mapsto & \mathcal{T}_\mathcal{A} := \{X \subseteq A | C_\mathcal{A}(X) = X\} \\ \mathcal{T}_\mathcal{A} & \mapsto & C_\mathcal{A}(X) := \bigcap \{T \in \mathcal{T}_\mathcal{A} | X \subseteq T\} \end{array}$$

Logics as coalgebras

Palmigiano shows in [Pal02]

- that an abstract logic can be represented by a coalgebra
- these coalgebras maps a formula into the set of its theories;
- the morphisms on that category correspond exactly to the usual morphisms between logics.
- the class of coalgebras that corresponds to abstract logics of empty signature defines a covariety.

Logics as coalgebras

closure system (contravariant) functor: is the functor that maps a set in the set of the closure systems over it and, each function $f : A \to B$, in the map

$$\begin{array}{llll} \mathcal{C}(f): & \mathcal{C}(B) & \to & \mathcal{C}(A) \\ & \mathcal{F} & \mapsto & \{f^{-1}[T] : T \in \mathcal{F}\}. \end{array}$$

Let $\mathcal{A} = \langle A, \mathcal{T}_{\mathcal{A}} \rangle$.

$$A \xrightarrow{\ \xi\ } \mathcal{C}(A) \qquad a \longmapsto \xi(a) = \{T \in \mathcal{T}_{\mathcal{A}} | a \in T\}$$

Coalg$(\mathcal{C})$:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \xi & & \downarrow \eta \\ \mathcal{C}(A) & \xleftarrow{\mathcal{C}(f)} & \mathcal{C}(B) \end{array}$$

Fact [Pal02]

f is a logical morphism between two abstract logics iff it is a morphism between its underlying coalgebras.

Objectives

Logical interpretation on software development

- We introduced in [MMB09a, MMB09b, MMB10] a formalization of refinement on algebraic specifications based on logical interpretations;
- The formalization is suitable to deal with data encapsulation, decomposition of operations in atomic transactions, and on the reuse of specifications;

Aims

- The work aims to frame logical interpretation on the "logics as coalgebras" perspective;
- formalize refinement via interpretation on this setting;

Refinement by interpretation [MMB09a, MMB09b]

Interpretation

$\tau : \mathrm{Fm}(\Sigma) \to \mathcal{P}(\mathrm{Fm}(\Sigma'))$ interprets *SP* if there is a specification *SP*′ under Σ' such that:

- $\forall \varphi \in \mathrm{Fm}(Sig(SP)), SP \models \varphi$ iff $SP' \models \tau(\varphi)$

SP′ is a refinement by the interpretation τ of *SP* if

- τ interprets *SP*
- $\forall \varphi \in \mathrm{Fm}(Sig(SP)), SP \models \varphi$ implies $SP' \models \tau(\varphi)$

Theorem (Characterization)

$SP \rightharpoondown_\tau SP'$ if there is an interpretation SP^0 of SP such that $SP^0 \rightsquigarrow SP'$.

Outline

Strict refinement revisited

Definition

Let $\mathcal{A} = \langle A, C_{\mathcal{A}} \rangle$ *and* $\mathcal{A}' = \langle A, C_{\mathcal{A}'} \rangle$ *be two abstract logics.* $\mathcal{A} \rightsquigarrow \mathcal{A}'$*, if for any* $X \cup \{x\} \in A$, $x \in C_{\mathcal{A}}(X) \Rightarrow x \in C_{\mathcal{A}'}(X)$.

Theorem

$$\mathcal{A} \rightsquigarrow \mathcal{A}' \textit{ iff } \mathcal{T}_{\mathcal{A}'} \subseteq \mathcal{T}_{\mathcal{A}}.$$

First intuition

$$\begin{array}{ccc} A & \xhookrightarrow{\;i\;} & A \\ \downarrow \xi & & \downarrow \xi' \\ \mathcal{C}(A) & \xleftarrow{\;\mathcal{C}(i)\;} & \mathcal{C}(A) \end{array}$$

However, this implies that $\mathcal{T}_{\mathcal{A}'} = \mathcal{T}_{\mathcal{A}}$ and we just need the first inclusion!

Definition (Forward morphism)

A forward morphism between $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ with respect to a pre-order $\sqsubseteq$, is a map $h : A \to B$ such that $\mathcal{C}h \circ \beta \circ h \sqsubseteq \alpha$.

Theorem

$\mathcal{A}'$ is a strict refinement of $\mathcal{A}$ iff the inclusion map is a forward morphism from $\langle A, \xi \rangle$ to $\langle A, \xi' \rangle$ wrt $\subseteq$.

Theorem

The tuple $\langle \mathbf{Log}, \mathbf{ref}, i, \circ \rangle$, where

- **Log** *is the class of $\mathcal{C}$-coalgebras induced by abstract logics;*
- **ref** *is the class of its inclusion forward morphisms wrt $\subseteq$;*
- *i is the class of identical maps;*
- *$\circ$ is the composition of functions,*

defines a category.

Relating logics: Morphisms & Interpretations

Definition (Logical morphism)

A logical morphism between the logics $\mathcal{A} = \langle A, \mathcal{T}_\mathcal{A} \rangle$ and $\mathcal{B} = \langle B, \mathcal{T}_\mathcal{B} \rangle$ consists of an (algebraic) morphism $h : A \rightarrow B$ such that

$$\{h^{-1}[T'] \| T' \in \mathcal{T}_\mathcal{B}\} = \mathcal{T}_\mathcal{A}.$$

Definition (Interpretation)

Let $\mathcal{A} = \langle A, C_\mathcal{A} \rangle$ and $\mathcal{B} = \langle B, C_\mathcal{B} \rangle$ be two abstract logics. A multifunction $f : A \rightrightarrows B$ is an interpretation ($f : \mathcal{A} \rightrightarrows \mathcal{B}$ for short), if for any $\{x\} \cup X \subseteq A$,

$$x \in C_\mathcal{A}(X) \Leftrightarrow f(x) \subseteq C_\mathcal{B}(f[X]).$$

Outline

Some preliminaries

Let $f : A \rightrightarrows B$ be a multifunction

- image: $f[X] = \bigcup\{f(a) | a \in X\}$;
- inverse image: $f^{-1}[Y] = \{a \in A | f(a) \subseteq Y\}$

Let $\mathcal{A} = \langle A, C_{\mathcal{A}} \rangle$ and $\mathcal{B} = \langle B, C_{\mathcal{B}} \rangle$ two abstract logics. The multifunction $f : A \rightrightarrows B$ is said to be

- continuous wrt $\mathcal{A}$ and $\mathcal{B}$ if for every $X \subseteq A$, $f[C_A(X)] \subseteq C_B(f[X])$
- closed if maps closed set wrt $\mathcal{A}$ in closed sets wrt $\mathcal{B}$;

The category of logics and interpretations

Definition (Interpretation)

Let $\mathcal{A} = \langle A, C_\mathcal{A} \rangle$ and $\mathcal{B} = \langle B, C_\mathcal{B} \rangle$ be two abstract logics. A multifunction $f : A \rightrightarrows B$ is an interpretation, if for any $\{x\} \cup X \subseteq A$,

$$x \in C_\mathcal{A}(X) \Leftrightarrow f(x) \subseteq C_\mathcal{B}(f[X]).$$

Lemma

f is an interpretation iff for any $X \subseteq A$, $C_\mathcal{A}(X) = f^{-1}[C_\mathcal{B}(f[X])]$.

Lemma

Let $\mathcal{A} = \langle A, C_\mathcal{A} \rangle$ and $\mathcal{B} = \langle B, C_\mathcal{B} \rangle$ be two abstract logics and $f : A \rightrightarrows B$ a closed and continuous multifunction wrt $\mathcal{A}$ and $\mathcal{B}$. TFAE:

1. *f is an interpretation from $\mathcal{A}$ into $\mathcal{B}$;*
2. *for any closed set T wrt $\mathcal{A}$, $T = f^{-1}[C_\mathcal{B}(f[T])]$.*

The category of logics and interpretations

Theorem

The tuple $\langle \mathbf{Log}, \mathbf{Int}, i, \circ \rangle$, *where*

- $\mathbf{Log}$ *is the class of abstract logics;*
- $\mathbf{Int}$ *is the class of its interpretations;*
- i *is the class of identical maps (for each abstract logic* $\langle A, C_{\mathcal{A}} \rangle$ *the identical map* $i_{\mathcal{A}} : A \rightrightarrows A$*);*
- $\circ$ *is the composition of multifunctions,*

defines a category.

Logic induced by the Frege relation

The abstract logic co-induced by f and $\mathcal{A}$ in B is defined as the abstract logic $\mathcal{B} = \langle B, C_f \rangle$, where C_f is such that $\mathrm{Th}\mathcal{B} = \{T | f^{-1}[T] \in \mathrm{Th}\mathcal{A}\}$

- Frege relation: $\sim_{\mathcal{A}} = \{\langle a, b \rangle \in A^2 | C_{\mathcal{A}}(a) = C_{\mathcal{A}}(b)\}$;
- Canonical epimorphism $e : A \rightrightarrows A/\sim$, such $e_\sim(a) = [a]_\sim$.
- $\mathcal{A}_\sim := \langle A/\sim, C_{e_\sim} \rangle$;

Lemma

For any abstract logic $\mathcal{A} = \langle A, C_{\mathcal{A}} \rangle$, the multifunction $e : A \rightrightarrows A_\sim$ is an interpretation from $\mathcal{A}$ to $\mathcal{A}_\sim$.

Theorem

Let $\mathcal{A} = \langle A, C_{\mathcal{A}} \rangle$ and $\mathcal{B} = \langle B, C_{\mathcal{B}} \rangle$ be two abstract logics. Then there exists an interpretation $f : A \rightrightarrows B$ iff there exists an interpretation $f^* : A_\sim \rightrightarrows B_\sim$.

Frame interpretation on the coalgebraic view

For $\mathcal{A} = \langle A, \mathcal{T}_{\mathcal{A}} \rangle$:

In [Pal02]:

$$a \overset{\eta}{\longmapsto} \eta(a) = \{T \in \mathcal{T}_{\mathcal{A}} | a \in T\}$$

Our aim:

$$X \overset{\xi}{\longmapsto} \xi(X) = \{T \in \mathcal{T}_{\mathcal{A}} | X \subseteq T\}$$

Frame interpretation on the coalgebraic view

Category **Pw**

Let **Pw** be the category with

- $Obj(\mathbf{Pw}) = \{\mathcal{P}(X) | X \in Obj(\mathbf{Set})\}$;
- $Arrow(\mathbf{Pw})$ are the functions between **Pw** objects.

$\bar{\mathcal{C}} : \mathbf{Pw} \longrightarrow \mathbf{Pw}$

$$\bar{\mathcal{C}}(X) := \{\mathcal{S} \subseteq X | \mathcal{S} \text{ is a closure system}\}$$

$$\begin{array}{rccc} \bar{\mathcal{C}}(f): & \bar{\mathcal{C}}(B) & \longrightarrow & \bar{\mathcal{C}}(A) \\ & \mathcal{F} & \mapsto & \{f^{-1}[T] : T \in \mathcal{F}\}. \end{array}$$

Power-function

A multifunction $f : A \rightrightarrows B$ induces a function

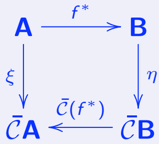
$$\begin{array}{rccc} f^*: & \mathcal{P}(A) & \longrightarrow & \mathcal{P}(B) \\ & X & \mapsto & \bigcup_{x \in X} f(x). \end{array}$$

Theorem

Let $\mathcal{A} = \langle A, \mathcal{T}_\mathcal{A} \rangle$ and $\mathcal{B} = \langle B, \mathcal{T}_\mathcal{B} \rangle$ be two abstract logics and $f : A \rightrightarrows B$ an interpretation. Then, $\mathcal{T}_\mathcal{A} = \{f^{-1}[T] : T \in \mathcal{T}_\mathcal{B}\}$.

Corollary

Let $\mathcal{A} = \langle A, C_\mathcal{A} \rangle$ and $\mathcal{B} = \langle B, C_\mathcal{B} \rangle$ be two abstract logics and $\langle A, \xi \rangle$, $\langle B, \eta \rangle$ the coalgebras induced by them. Hence, if $f : A \rightrightarrows B$ is an interpretation, then f^* is a coalgebraic morphism between its logics, i.e., f^* makes the following diagram to commute:

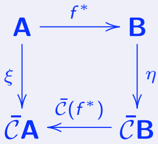


Theorem

Let $\mathcal{A} = \langle A, \mathcal{T}_{\mathcal{A}} \rangle$ and $\mathcal{B} = \langle B, \mathcal{T}_{\mathcal{B}} \rangle$ be two abstract logics and $f : A \rightrightarrows B$ an interpretation. Then, $\mathcal{T}_{\mathcal{A}} = \{f^{-1}[T] : T \in \mathcal{T}_{\mathcal{B}}\}$.

Corollary

Let $\mathcal{A} = \langle A, C_{\mathcal{A}} \rangle$ and $\mathcal{B} = \langle B, C_{\mathcal{B}} \rangle$ be two abstract logics and $\langle A, \xi \rangle$, $\langle B, \eta \rangle$ the coalgebras induced by them. Hence, if $f : A \rightrightarrows B$ is an interpretation, then f^* is a coalgebraic morphism between its logics, i.e., f^* makes the following diagram to commute:



Theorem

Let $\mathcal{A} = \langle A, C_A \rangle$ and $\mathcal{B} = \langle B, C_B \rangle$ be two abstract logics and $f : A \rightrightarrows B$ a closed and continuous multifunction. Then, $\mathcal{T}^A = \{f^{-1}[T] : T \in \mathcal{T}^B\}$ implies that f is an interpretation.

Strict refinement

Theorem (Characterization)

$SP \rightharpoondown_\tau SP'$ if there is an interpretation SP^0 of SP such that $SP^0 \rightsquigarrow SP'$.

Strict refinements on **Pw**

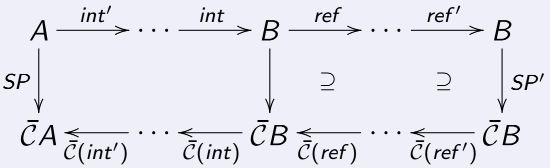
$$\begin{array}{ccc} A & \overset{i}{\hookrightarrow} & A \\ \xi \downarrow & \supseteq & \downarrow \eta \\ \mathcal{C}(A) & \underset{\mathcal{C}(i)}{\leftarrow} & \mathcal{C}(A) \end{array} \quad \text{corresponds to} \quad \begin{array}{ccc} \mathbf{A} & \overset{i^*}{\hookrightarrow} & \mathbf{A} \\ \xi' \downarrow & \supseteq & \downarrow \eta' \\ \bar{\mathcal{C}}(\mathbf{A}) & \underset{\mathcal{C}(i^*)}{\leftarrow} & \bar{\mathcal{C}}(\mathbf{A}) \end{array}$$

for $\mathbf{A} = \mathcal{P}(A)$

Refinement via interpretation

Theorem (Characterization)

$SP \rightharpoondown_{\tau} SP'$ if there is an interpretation SP^0 of SP such that $SP^0 \rightsquigarrow SP'$.



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Conclusions

- We generalize the coalgebraic perspective of logics presented in [Pal02], capturing the interpretations of logics with coalgebraic morphisms ;
- taking this approach, we present an elegant formalization of the refinement via interpretation concept;

Directions to pursue

- An interpretation entails the existence of a bisimilation; what is the logical counterpart to the existence of $\langle \xi, \eta \rangle$-bisimilation?
  - rephrase this work in the relational setting.
- explore in the "logics as coalgebras" perspective
  - finitarity: $C(X) = \{C(Y) : Y \subseteq X, Y \text{ finite}\}$
  - structurality: by considering the algebraic structure on underlying sets of the logics.

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