Algebraically Enriched Coalgebras

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Motivation

- One of the nice things about (modelling systems as) coalgebras:

  The type of the system determines a canonical notion of equivalence.

  e.g bisimilarity for LTS's

- One of the not so nice things about coalgebras:

  The canonical notion of equivalence is not what one wants.

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  Goal of this talk: Show a way of uniformly deriving a new set of canonical equivalences from the type of the system.

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Example I: Determinizing (coalgebraically)

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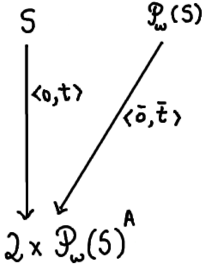

$$\overline{o}(Q) = \begin{cases} 1 & \exists_{q \in Q} o(q) = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \overline{t}(Q)(a) = \bigcup_{q \in Q} t(q)(a)$$

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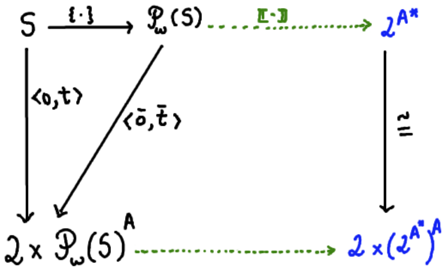

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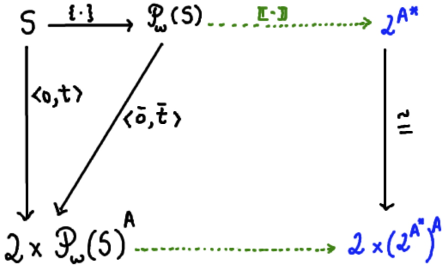

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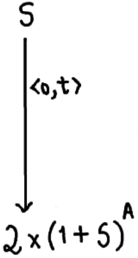
How do we study NDA wrt language equivalence?

$$L_s = [\![\{s\}]\!]$$

Example II: Totalizing



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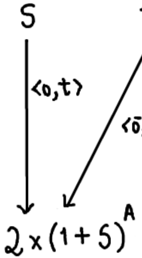

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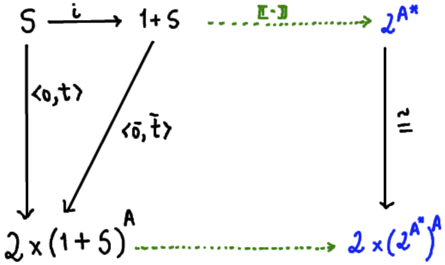

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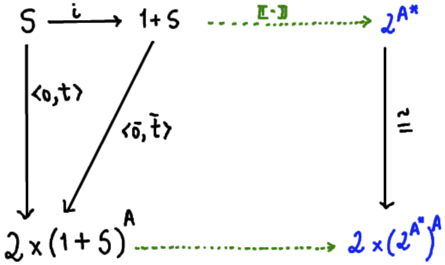

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How do we study PA wrt language equivalence?

$$L_s = [\![\, i(s)\,]\!]$$

Example III: Linearization

$$\begin{array}{c} S \\ \Big\downarrow \langle o,t \rangle \\ \mathbb{R} \times (\mathbb{R}_\omega^S)^A \end{array}$$

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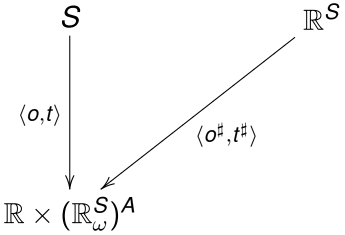

$$o^\sharp\left(\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}\right) = \sum v_i \times o(s_i) \qquad t^\sharp\left(\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}\right)(a)(s_j) = \sum v_i \times t(s_i)(a)(s_j)$$

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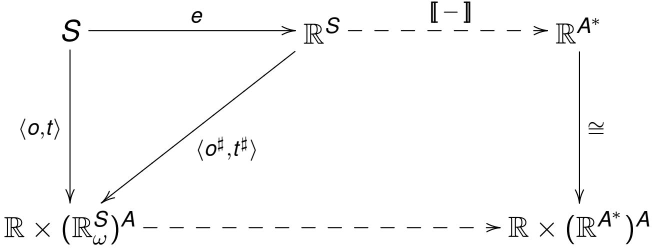

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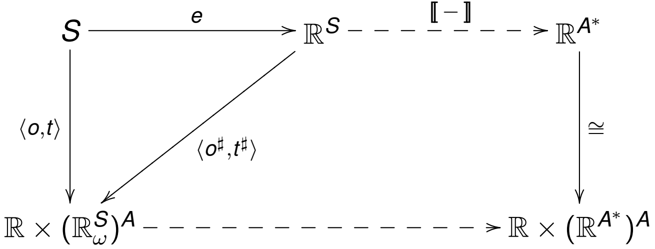

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How do we study WA wrt weighted languages (linear bisimilarity)?

$$L_s = [\![\, e(s) \,]\!]$$

Chasing the pattern. . .

How do we capture all the examples (and more) in the same framework?

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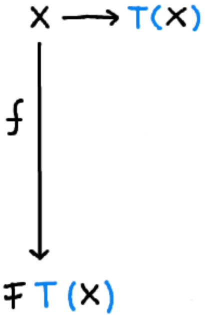

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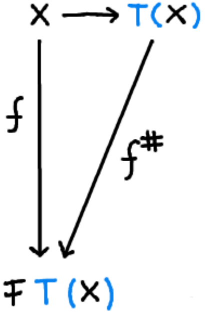

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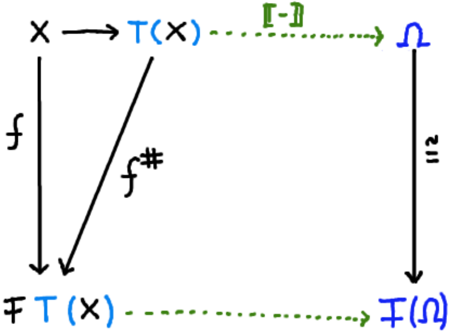

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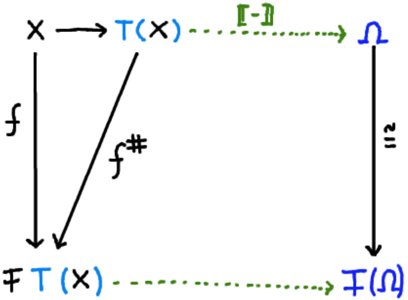
Transform an FT-coalgebra (X,f) into an F-coalgebra $(T(X), f^\sharp)$.
If F has final coalgebra: $x_1 \approx_F^T x_2 \Leftrightarrow [\![\eta_X(x_1)]\!] = [\![\eta_X(x_2)]\!]$.

In a nutshell. . .



Ingredients:

- A monad T;
- A final coalgebra for F (for instance, take F to be bounded);
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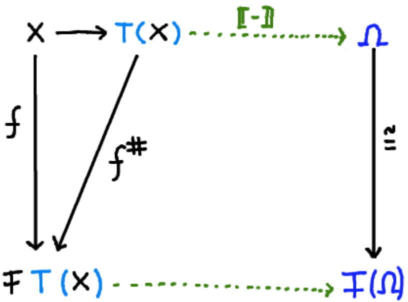

Ingredients:

- A monad T;
- A final coalgebra for F (for instance, take F to be bounded);
- An extension $f^\sharp$ of f; We can require $FT(X)$ to be a T-algebra: $(FT(X), h\colon T(FT(X)) \to FT(X))$

$$f^\sharp\colon T(X) \xrightarrow{T(f)} T(F(T(X))) \xrightarrow{h} F(T(X))$$

Bisimilarity implies T-enriched bisimilarity

Theorem

$$\sim_{FT} \quad \Rightarrow \quad \approx_F^T$$

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The above theorem instantiates to well known facts:

- for NDA ($F(X) = 2 \times X^A$, $T = \mathcal{P}$) that bisimilarity implies language equivalence;
- for PA ($F(X) = 2 \times X^A$, $T = 1 + -$) that equivalences of pair of languages, consisting of defined paths and accepted words, implies equivalence of accepted words;
- for weighted automata ($F(X) = \mathbb{R} \times X^A$, $T = \mathbb{R}_\omega^-$) that weighted bisimilarity implies weighted language equivalence.

Examples, Examples, Examples,. . .

- **Partial Mealy machines** $S \to (B \times (1+S))^A$;
- **Automata with exceptions** $S \to 2 \times (E+S)^A$;
- **Automata with side effects** $S \to E^E \times ((E \times S)^E)^A$;
- **Total subsequential transducers** $S \to O^* \times (O^* \times S)^A$;
- **Probabilistic automata** $S \to [0, 1] \times (\mathcal{D}_\omega(X))^A$;
- . . .

Conclusions

- Lifted *powerset construction* to the more general framework of FT-coalgebras;
- Uniform treatment of several types of automata, recovery of known constructions/results;
- Opens the door to the study of T-*enriched equivalences* for many types of automata.

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Thanks!!

The relation with [HJS]

1. Some examples do not fit their framework (e.g., interactive output monad is not commutative, side-effect monad has no $\bot$,...); some of our examples might not fit our framework (?);
2. If $FT \cong TG$ (e.g $2 \times \mathcal{P}(-)^A \cong \mathcal{P}(1 + A \times -)$) then:

$$x \sim_{tr} y \iff x \approx_F^T y$$

If $\rho\colon TG \Rightarrow FT$ then:

$$x \sim_{tr} y \Rightarrow x \approx_F^T y$$