Similarity quotients as final coalgebras

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Outline

1. Examples
2. General Theory

Examples

We study the following examples:

1. bisimilarity
2. bisimilarity and similarity together
3. similarity
4. upper similarity
5. intersection of lower and upper similarity
6. 2-nested similarity

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6. 2-nested similarity

In each case we see

- how to use a final coalgebra
- how to construct a final coalgebra.

Bisimilarity: Using A Final Coalgebra

Fix a countable set Act of labels.

Let $F : X \mapsto \mathcal{P}^{\leqslant \aleph_0}(\mathrm{Act} \times X)$ on **Set**.

A countably branching Act-labelled transition system is an F-coalgebra.

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Theorem: no junk

Every element of A is of the form $\sigma_B b$.

Bisimilarity: Constructing A Final Coalgebra

Let $F : X \mapsto \mathcal{P}^{\leqslant\aleph_0}(\mathrm{Act} \times X)$ on **Set**.

Suppose A is a transition system that is big enough
i.e. every $b \in B$ is bisimilar to some $a \in A$.

Then A modulo bisimilarity (with behaviour map chosen to make $A \longrightarrow A/\simeq$ a homomorphism) is a final F-coalgebra.

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Example of a big enough transition system

$A \stackrel{\mathrm{def}}{=}$ the disjoint union of all transition systems on initial segments of $\mathbb{N}$.
It's big enough because every (B, b) has countably many successors.

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If A isn't big enough, then $A/\simeq$ is still subfinal, i.e. parallel morphisms to it are equal.

Bisimilarity and Similarity: Using A Final Coalgebra

Let G be the endofunctor on **Preord** mapping $(X, \leqslant)$ to

$$(\mathcal{P}^{\leqslant\aleph_0}(\mathsf{Act} \times X), \mathsf{Sim}(\leqslant))$$

where $U\ \mathsf{Sim}(\mathcal{R})\ V \stackrel{\text{def}}{\Leftrightarrow} \forall x \in U. \exists y \in V. u\ \mathcal{R}\ v$.

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Bisimilarity and similarity: constructing a final coalgebra (Hughes-Jacobs)

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Then A modulo bisimilarity, preordered by similarity, is a final G-coalgebra.

Quotienting by a preorder

If A is a set with an equivalence relation $\sim$
then $A/\sim$ is a set consisting of the equivalence classes

$$[a]_\sim \stackrel{\text{def}}{=} \{x \in A \mid x \sim a\}$$

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So **Poset** is a full reflective subcategory of **Preord**.

$$\mathbf{Poset} \underset{\hookrightarrow}{\overset{Q}{\underset{\perp}{\longleftarrow}}} \mathbf{Preord}$$

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Then A modulo similarity is a final H-coalgebra.

Lower and Upper Simulations (Ulidowski, Lassen, Pitcher)

A countably branching Act-labelled transition system with divergence is a coalgebra for $X \mapsto \mathcal{P}^{\leqslant\aleph_0}((\mathsf{Act} \times X)_\perp)$.

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Let B and C be two such, and let $\mathcal{R} \subseteq B \times C$ be a relation.

Lower simulation

$\mathcal{R}$ is a lower simulation when, for $b \ \mathcal{R}\ c$

- $b \overset{a}{\rightsquigarrow} b'$ implies there is c' such that $c \overset{a}{\rightsquigarrow} c'$ and $b' \ \mathcal{R}\ c'$.

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There are many variants.

Upper similarity and final coalgebras

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$$U \; \mathsf{Upper}(\mathcal{R}) \; V \quad \overset{\text{def}}{\Leftrightarrow} \quad \perp \notin U \Rightarrow \\ (\perp \notin V \wedge \\ \forall y \in V.\, \exists x \in U.\, x \,\mathcal{R}\, y)$$

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Then

- a final H-coalgebra characterizes upper similarity, with no junk
- a big enough transition system with divergence, modulo upper similarity, gives a final H-coalgebra.

Doubly preordered sets

We are going to study the intersection of lower similarity and upper similarity.

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We obtain categories

$$\mathbf{TwoPoset} \underset{\hookrightarrow}{\overset{Q}{\underset{\perp}{\longleftarrow}}} \mathbf{TwoPreord}$$

Lower similarity $\cap$ upper similarity

Let G be the endofunctor on **TwoPreord** mapping $(X, \leqslant_l, \leqslant_u)$ to

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Lower similarity ∩ upper similarity

Let *G* be the endofunctor on **TwoPreord** mapping $(X, \leqslant_l, \leqslant_u)$ to

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2-nested simulation (Groote and Vaandrager)

Let B and C be transition systems.

A 2-nested simulation from B to C is a simulation contained in the converse of a simulation.
Equivalently a simulation contained in the converse of similarity.
Equivalently a simulation contained in mutual similarity.

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Corresponds to modal formulas $\Diamond^n$ and $\Diamond^n \Box^m$.

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Then

- a final H-coalgebra characterizes (the converse of) similarity and 2-nested similarity, with no junk
- a big enough transition system, modulo 2-nested similarity, gives a final H-coalgebra.

Proving these results simultaneously

Instead of having to prove all these results separately,

we want general theorems that they are all instances of.

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What is the data for our theorems?

The two categories

We want two categories with the same objects:

- a regular category $\mathcal{C}$ of functions
- a category $\mathcal{A}$ of quasi-relations

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Examples with $\mathcal{C} = \mathbf{Set}$

A quasi-relation $A \xrightarrow{\mathcal{R}} B$ is

1. a relation
2. a pair of relations $(\mathcal{R}_l, \mathcal{R}_u)$
3. a pair of relations $(\mathcal{R}_1, \mathcal{R}_2)$ with $\mathcal{R}_2 \subseteq \mathcal{R}_1$.

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We write $P(X)$ for the quasi-predicates on X, given by a regular fibration on $\mathcal{C}$, then define

$$\mathcal{A}(X, Y) \stackrel{\text{def}}{=} P(X \times Y)$$

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(For well-pointed $\mathcal{C}$: when the preorder on $\mathcal{C}(1, X)$ is antisymmetric.)

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- quotiented
- augmented by an arbitrary $X \xrightarrow{\mathcal{R}} X$.

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We obtain an adjunction

$$\mathbf{QPoset} \underset{\longleftarrow}{\overset{Q}{\longleftarrow}} \perp \hookrightarrow \mathbf{QPreord}$$

Relational extension

We require an endofunctor F on $\mathcal{C}$, expressing the behaviour

Examples with $\mathcal{C} = \mathbf{Set}$

$$\begin{array}{lcl} X & \mapsto & \mathcal{P}^{\leqslant \aleph_0}(\mathsf{Act} \times X) \\ X & \mapsto & \mathcal{P}^{\leqslant \aleph_0}((\mathsf{Act} \times X)_\bot) \end{array}$$

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And we need Γ mapping a quasi-relation $X \xrightarrow{\mathcal{R}} Y$ to $FX \xrightarrow{\Gamma\mathcal{R}} FY$.

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And we need Γ mapping a quasi-relation $X \xrightarrow{\mathcal{R}} Y$ to $FX \xrightarrow{\Gamma\mathcal{R}} FY$.

Γ is a relational extension of F.

Properties of relational extension (1)

Monotonicity

$$X \xrightarrow{\mathcal{R},\mathcal{R}'} Y$$

$$\mathcal{R} \subseteq \mathcal{R}' \Rightarrow \Gamma\mathcal{R} \subseteq \Gamma\mathcal{R}'$$

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Stability (Hughes and Jacobs)

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ & & \big\downarrow \mathcal{R} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

$$\Gamma((f \times g)^{-1}\mathcal{R}) = (Ff \times Fg)^{-1}\mathcal{R}$$

Properties of relational extension (2)

Lax functoriality

$$\begin{array}{rcl} \mathsf{id}_{\Gamma X} & \subseteq & \Gamma \mathsf{id}_X \\ (\Gamma\mathcal{R});(\Gamma\mathcal{S}) & \subseteq & \Gamma(\mathcal{R};\mathcal{S}) \end{array}$$

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But that excludes the case of 2-nested simulation.

Simulation on transition systems

Let (A, ζ) and (B, ζ') be F-coalgebras ("transition systems").

A simulation from (A, ζ) to (B, ζ') is a quasi-relation $A \xrightarrow{\mathcal{R}} B$ such that $\mathcal{R} \subseteq (\zeta \times \zeta')^{-1} \Gamma \mathcal{R}$.

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This property is preserved by composition (and identity), and by pullback.

Ordered Coalgebras

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So H-coalgebras are a generalization of "transition systems" (F-coalgebras).

We can define a notion of simulation on these too.

Using A Final Coalgebra: The Greatest Simulation

Suppose we have a final H-coalgebra $(X, (\leqslant), \zeta)$.

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Using A Final Coalgebra: No junk

Some categories have the property that all regular epimorphisms split.

Example **Set** (Axiom of Choice)
Example $\mathbf{Set}^I$ for any set I
Non-example $\mathbf{Set}^{\cdot\to\cdot}$

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No junk theorem, formal version

There is $X \xrightarrow{\xi} FX$ such that the anamorphism $\Delta(X, \xi) \xrightarrow{a} A$ is just id_X.

No junk theorem, informal version

Every element of A is the anamorphic image of some node of an ordinary transition system.

Quotienting By Similarity

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And QA is a sub-final H-coalgebra.

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Informal definition

every node of a is mapped to a node that is mutually similar to it.

Formal definition

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Then QA is final.

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Then $Q\Delta A$ is a final H-coalgebra.

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We have a general theory giving various branching-time relations in terms of final coalgebras, and vice versa.

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This covers all our examples of endofunctors on **Poset**, **TwoPoset**, **NestPoset**.

To incorporate the examples of endofunctors on **Set**, we add an extra parameter to the theory.

New notion of relational extension that includes 2-nested similarity.

Question

Metric spaces as an example?