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Conway Games, algebraically and coalgebraically

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Introduction

- games arising in real life are extremely varied
- a wide gamut of interactions and other dynamic phenomena are described using game-based metaphors
- many concepts, no universal meaning: move, position, play, turn, winning condition, payoff function, strategy, ...
- Conway's games: very elementary but structured, sufficiently abstract notion of game
- other notions of games can be encoded, e.g. automata games
- algebraic-coalgebraic methods provide a convenient conceptual setting

Finite vs Infinite Plays

- In [“On Numbers and Games”] Conway focuses mainly on finite, i.e. terminating games. Infinite games are neglected as ill-formed or trivial, not interesting for “busy men”;
- however, especially in view of applications, potentially infinite interactions are even more important than finite ones.
- In [CALCO’09]:
  - a theory of infinite games (hypergames) is studied in a coalgebraic (coinductive) setting;
  - infinite plays are considered as draws;
  - the notion of winning strategy is replaced by that of non-losing strategy;
  - the theory on hypergames extends that of Conway’s games.

Further developments

In the present talk, we will focus on:

equivalences and congruences on games and hypergames.

Most results on games and hypergames can be understood in terms of equivalences.

Classical combinatorial games

- 2-player games, Left (L) and Right (R)
- games have positions
- L and R move in turn
- perfect knowledge: all positions are public to both players
- in any position there are rules which restrict L to move to any of certain positions (Left positions), while R may similarly move only to certain positions (Right positions)
- the game ends when one of the two players does not have any option

Many Games played on boards are combinatorial games: Nim, Domineering, Go, Chess.

Conway Games, algebraic definition

Games are identified with initial positions.
Any position p is determined by its Left and Right options, $p = (P^L, P^R)$.

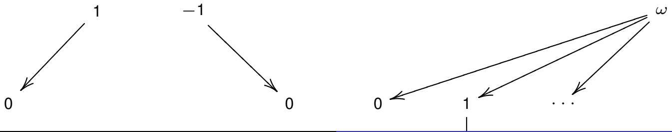
The class $\mathcal{G}$ of games is inductively defined by:

- the empty game $(\{\}, \{\}) \in \mathcal{G}$;
- if $P, P' \subseteq \mathcal{G}$, then $(P, P') \in \mathcal{G}$.

$\mathcal{G}$ is the carrier of the initial algebra $(\mathcal{G}, id)$ of the functor $F : \mathcal{C} \rightarrow \mathcal{C}$,

$$F(X) = \mathcal{P}(X) \times \mathcal{P}(X) \quad F(f) = \mathcal{P}(f) \times \mathcal{P}(f)$$

$\mathcal{C}$ is the category of classes (of hyper(sets) or sets with hereditarily cardinal less than κ).



Winning Strategies for L, R, I, II

L Left Player
R Right Player
I player: the player who starts the game
II player: the player who responds to the I player

- winning condition for a player: no more moves for the other player
- a winning strategy for L (R) player tells, at each step, which is the next L (R) move, in response to any possible last move of R (L), independently whether L (R) acts as I or II player
- a winning strategy for I (II) player tells, at each step, which is the next move of the I (II) player, in response to any possible last move of the II (I) player, independently whether I (II) acts as L or R player
- winning strategies are positional (history-free)
- winning strategies are formalized as partial functions from positions to moves

Combining games: Conway's sum

On the sum game, at each step, the current player chooses one component game and performs a move on that component

$$x + y = (\{x^L + y \mid x^L \in X^L\} \cup \{x + y^L \mid y^L \in Y^L\},\ \{x^R + y \mid x^R \in X^R\} \cup \{x + y^R \mid y^R \in Y^R\})\ .$$

- Any player can change the component.
- On a sum game we loose the alternation of I and II players in the single components. This is why we need to distinguish also between L and R player.

Equivalences and Congruences on Games

- We focus on the subclass of impartial games, where L and R have the same options.
- Thus we can consider only I and II player.
- Impartial games can be represented by $x = X$.
- They form an algebra $\mathcal{I} = \mathcal{P}(\mathcal{I})$.

- Conway equivalence on surreal numbers:

$$x \sim y \ \text{ iff } \ \forall x' \in X.\ (y \not\sim x') \ \wedge \ \forall y' \in Y.\ (y' \not\sim x)\ .$$

Lemma: $x \sim y$ iff $x + y$ has a winning strategy for II.

- Contextual equivalence:

$$x \approx y \iff \forall C[\,].\ C[x] \Updownarrow C[y]\ ,$$

where

  - $x \Updownarrow y$ iff whenever there is a winning strategy for I (II) on x there is also one on y, and vice versa.
  - additive contexts:

$$C[\,] \ ::= \ [\,] \mid C[\,] + x \mid x + C[\,]$$

Lemma:
i) $\approx$ is the greatest congruence included in $\Updownarrow$.
ii) the class of additive contexts can be simplified:

$$x \approx y \text{ iff } \forall z.\ x + z \Updownarrow y + z\ .$$

Theorem: $\sim \ = \ \approx$

Grundy-Sprague Semantics

There exists a system of canonical games $\{*\alpha\}_{\alpha \in Ord}$

$$*\alpha = \{*\beta \mid \beta < \alpha\} \quad \text{(Nim games)}$$

The Grundy function $g : \mathcal{I} \to Ord$ associates to each impartial game x an ordinal α such that $x \approx *g(x)$.
g is computed on the game graph using the mex (minimal excludent) algorithm.

The Grundy semantics is

- compositional w.r.t. sum
- fully abstract w.r.t. $\approx$, i.e.: $g(x) = g(y)$ iff $x \approx y$.

A categorical representation of the equivalence: Joyal category of games

- objects: (impartial) games

- morphisms:
$f : x \to y$ winning strategy for II on $x + y$ $\longleftrightarrow$ $x \approx y$

- identity: copy-cat strategy $\longleftrightarrow$ reflexivity

- composition:
via the swivel chair strategy (trace operator) $\longleftrightarrow$ transitivity

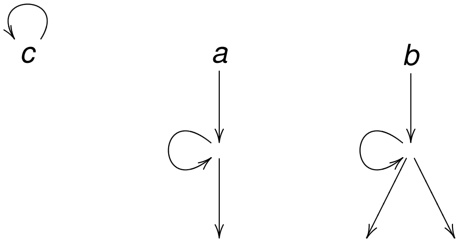
- + symmetric monoidal functor $\longleftrightarrow$ congruence

Hypergames and non-losing Strategies

Hypergames $\mathcal{H}$ are the carrier of the final coalgebra $(\mathcal{H}, id)$ of the functor $F\mathcal{C} \to \mathcal{C}$, $FX = \mathcal{P}(X) \times \mathcal{P}(X)$.

Plays on hypergames can be non-terminating.
A non-terminating play is a draw.
The notion of winning strategy is replaced by that of non-losing strategy.

Impartial hypergames are the carrier $\mathcal{J}$ of the final coalgebra of $\mathcal{P}$.



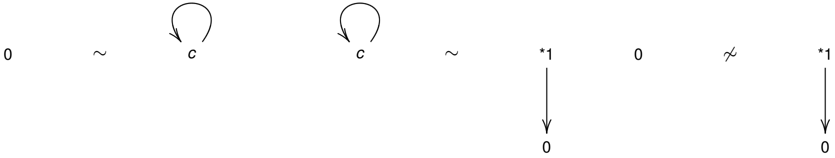
Extending Conway's equivalence on surreal numbers to hypergames

It requires a simultaneous coinductive definition for defining both relations $\sim$ and $\not\sim$, as the greatest fixpoint of the monotone operator $\Phi : \mathcal{P}(\mathcal{H} \times \mathcal{H}) \times \mathcal{P}(\mathcal{H} \times \mathcal{H}) \longrightarrow \mathcal{P}(\mathcal{H} \times \mathcal{H}) \times \mathcal{P}(\mathcal{H} \times \mathcal{H})$

$$\Phi(\mathcal{R}_1, \mathcal{R}_2) = (\{(x, y) \mid \forall x' \in X. y\mathcal{R}_2 x' \ \wedge \ \forall y' \in Y. y'\mathcal{R}_2 x\}, \\ \{(x, y) \mid \exists x' \in X. y\mathcal{R}_1 x' \ \vee \ \exists y' \in Y. y'\mathcal{R}_1 x\})$$

Lemma: $x{\sim}y$ iff $x + y$ has a non-losing strategy for II.

But: $\sim$ is not transitive.



Contextual equivalence and extended Grundy semantics

- Contextual equivalence:

$$x \approx y \iff \forall C[\,].\ C[x] \Updownarrow C[y]\ ,$$

  where
  $x \Updownarrow y$ iff whenever there is a non-losing strategy for I (II) on x there is also one on y, and vice versa.

- There is a system of canonical hypergames extending canonical games with hypergames $*\infty_K$, where

$$*\infty_\emptyset = \{*\infty_\emptyset\}$$

$$*\infty_K = \{*\infty_\emptyset\} \cup \{*k \mid k \in K\}$$

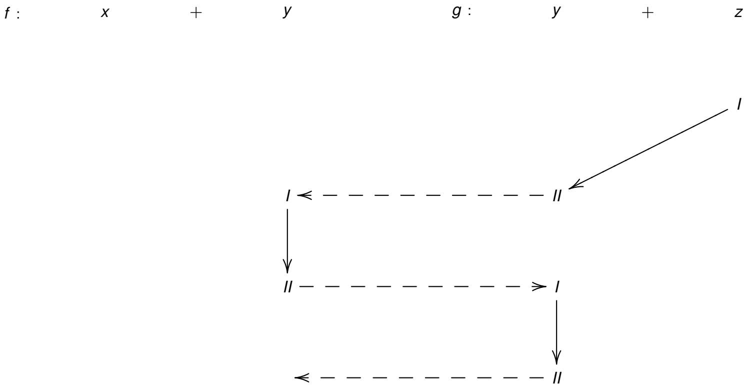
- The Grundy semantics can be extended to hypergames $\gamma : \mathcal{J} \to Ord \cup \{\infty_K \mid K \subseteq Ord\}$

Theorem: γ is fully abstract w.r.t. $\approx$.

Categories of hypergames: a first try

A morhism $f : x \to y$ is a non-losing strategy for II on $x + y$.

This captures $\sim$ on hypergames, which is not transitive,
hence no closure under composition,
namely the swivel chair strategy contains an infinite play



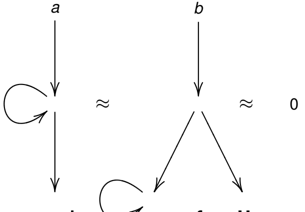
A category of balanced strategies

To avoid infinite plays in the swivel chair strategy, we introduce the notion of non-losing balanced strategy.

A morphism $f : x \to y$ is a non-losing balanced strategy on $x + y$, i.e. a non-losing strategy not containing plays which are definitely all in x or in y.

Non-losing balanced strategies are closed under composition and give rise to a category.

But: which notion of equivalence (congruence) do they capture?
Surprisingly, this is not $\approx$.



But: $a + b$ has no non-losing balanced strategy for II.

Questions

- Is there a notion of strategy/category capturing $\approx$?
- On the other perspective: what kind of contextual equivalence is captured by the category of balanced strategies?
- Can we tell apart *a* and *b*, by extending the class of additive contexts?
  - E.g. $C[\,] = \{\,\}$, $C[x] = \{x\}$. Does it give a finer equivalence? No, because the Grundy function is compositional also w.r.t. $C[\,]$.
  - We shall look for intensional contexts.
- What about a different notion of sum in the category?