Strong relative monads

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Motivation and contribution

- In programming language theory, we use structures like (strong) monads, (monoidal) comonads, arrows to structure syntax and semantics.
- Some natural structures fail to be monads as if for the only reason that the underlying functor is not an endofunctor.
- E.g., untyped/typed lambda calculus syntax (over finite contexts), finite-dimensional vector spaces etc.
- In FoSSaCS 2010, we defined and studied a relative monads as a generalization of monads.
- Here: strong relative monads.

Relative monads

- Given a category $\mathbb{C}$ and another category $\mathbb{J}$ with a functor $J \in \mathbb{J} \to \mathbb{C}$.
- A *relative monad* is given by
  - an object function $T \in |\mathbb{J}| \to |\mathbb{C}|$,
  - for any object $X \in |\mathbb{J}|$, a map $\eta_X \in \mathbb{C}(J\,X, T\,X)$ (unit),
  - for any objects $X, Y \in |\mathbb{J}|$ and map $k \in \mathbb{C}(J\,X, T\,Y)$, a map $k^* \in \mathbb{C}(T\,X, T\,Y)$ (Kleisli extension)

  satisfying

  - for any $X, Y \in |\mathbb{J}|$, $k \in \mathbb{C}(J\,X, T\,Y)$, $k^* \circ \eta_X = k$,
  - for any $X \in |\mathbb{J}|$, $\eta_X^* = \mathrm{id}_{T\,X} \in \mathbb{C}(T\,X, T\,X)$,
  - for any $X, Y, Z \in |\mathbb{J}|$, $k \in \mathbb{C}(J\,X, T\,Y)$, $\ell \in \mathbb{C}(J\,Y, T\,Z)$, $(\ell^* \circ k)^* = \ell^* \circ k^* \in \mathbb{C}(T\,X, T\,Z)$.
- T is functorial with $T\,f = (\eta \circ J\,f)^*$; η and $(-)^*$ are natural.

Relative monads (ctd)

- Ordinary monads arise as the special case where $\mathbb{J} =_{\mathrm{df}} \mathbb{C}$, $J =_{\mathrm{df}} \mathsf{Id}_{\mathbb{C}}$.
- Can define relative adjunctions between $J \in \mathbb{J} \to \mathbb{C}$ and $\mathbb{D}$.
- Every relative adjunction gives rise to a relative monad.
- Every relative monad resolves into a relative adjunction in at least two ways, the Kleisli and E-M adjunctions, which are its initial and final resolutions.
- If $\mathrm{Lan}_J \in [\mathbb{J}, \mathbb{C}] \to [\mathbb{C}, \mathbb{C}]$ exists, then $[\mathbb{J}, \mathbb{C}]$ has a lax monoidal structure and a relative monad on J is a lax monoid in it.
- If further conditions on J hold (in particular, J is fully faithful), then $[\mathbb{J}, \mathbb{C}]$ is (properly) monoidal and a relative monad on J is a (proper) monoid in it.

Example

- Given a semiring $(R, 0, +, 1, \times)$.
- Let $\mathbb{J} =_{\mathrm{df}} \mathbb{F}$, $\mathbb{C} =_{\mathrm{df}}$ **Set**, $J =$ the inclusion.
- Define
  - a set mapping $T \in \mathbb{F} \to$ **Set** by $T\ m =_{\mathrm{df}} J\ m \to R$,
  - for any $m \in |\mathbb{F}|$, a function $\eta_m \in J\ m \to T\ m$ by $\eta_m\,(i \in m) =_{\mathrm{df}} \lambda j \in m.\ \mathrm{if}\ i = j\ \mathrm{then}\ 1\ \mathrm{else}\ 0$
  - for any $m, n \in |\mathbb{F}|$, $A \in J\ m \to T\ n$, a function $A^* \in T\ m \to T\ n$ by $A^*\ x =_{\mathrm{df}} \lambda j \in n. \sum_{i \in m} x\ i \times A\ i\ j$

  $T\ \ m$ is the space of m-dimensional vectors, η_m is the diagonal $(m \times m)$-matrix, and $A^*\ x$ is the product of matrix A with a vector x.
- $(T, \eta, (-)^*)$ is a relative monad.
- $\mathrm{Kl}(T)$ is the category of finite-dimensional vector spaces and linear transformations.

Weak arrows

- Given a category $\mathbb{J}$, a *weak arrow* on $\mathbb{J}$ is given by
  - an object function $R \in |\mathbb{J}| \times |\mathbb{J}| \to \mathbf{Set}$,
  - for any objects $X, Y \in |\mathbb{J}|$, a function $\mathsf{pure} \in \mathbb{J}(X, Y) \to R(X, Y)$,
  - for any $X, Y, Z \in |\mathbb{J}|$, a function $(\lll) \in R(Y, Z) \times R(X, Y) \to R(X, Z)$

  satisfying
  - $\mathsf{pure}\,(g \circ f) = \mathsf{pure}\,g \lll \mathsf{pure}\,f$,
  - $r \lll \mathsf{pure}\,\mathsf{id} = r$,
  - $\mathsf{pure}\,\mathsf{id} \lll r = r$,
  - $t \lll (s \lll r) = (t \lll s) \lll r$.
- R extends to a functor $\mathbb{J}^{\mathrm{op}} \times \mathbb{J} \to \mathbf{Set}$ (an endoprofunctor on $\mathbb{J}$); pure and $\lll$ are natural.

Weak arrows = relative monads on Yoneda

- Assume $\mathbb{J}$ is small. Let $\mathbb{C} =_{\mathrm{df}} [\mathbb{J}^{\mathrm{op}}, \mathbf{Set}]$, $J = \mathbf{Y}$ (the Yoneda embedding).
- A weak arrow on $\mathbb{J}$ is a functor $R \in \mathbb{J}^{\mathrm{op}} \times \mathbb{J} \to \mathbf{Set}$ with structure.
- This is the same as a functor $T \in \mathbb{J} \to [\mathbb{J}^{\mathrm{op}}, \mathbf{Set}]$ with structure, in fact, a relative monad on $\mathbf{Y}$.

Monads vs relative monads

- Given any $\mathbb{C}, \mathbb{J}$ and $J \in \mathbb{J} \to \mathbb{C}$.
- If T is a monad on $\mathbb{C}$, then $T^\flat =_{\mathrm{df}} T \cdot J$ is a relative monad on $\mathbb{J}$.
- If J is well-behaved, then
  If T is a relative monad on J, then $T^\sharp =_{\mathrm{df}} \mathrm{Lan}_J\, T$ is a monad on $\mathbb{C}$.
- The adjunction

$$[\mathbb{C}, \mathbb{C}] \underset{\mathrm{Lan}_J}{\overset{- \cdot J}{\rightleftarrows}} [\mathbb{J}, \mathbb{C}] \qquad (\top)$$

lifts to an adjunction (a coreflection, if we require that J is fully-faithful)

$$\mathbf{Mnd}(\mathbb{C}) \underset{(-)^\sharp}{\overset{(-)^\flat}{\rightleftarrows}} \mathbf{Mnd}(J) \qquad (\top)$$

Strong relative monads

- Given a monoidal categories $(\mathbb{J}, I, \otimes)$, $(\mathbb{C}, I', \otimes')$ and a monoidal functor (J, e, m) between them.
- A strong relative monad is a relative monad $(T, \eta, (-)^*)$ and, for any $X, Y \in |\mathbb{J}|$, a map $\mathrm{st}_{X,Y} \in \mathbb{C}(TX \otimes' JY, T(X \otimes Y))$, natural in X, Y, with $T, \eta, (-)^*$ strong wrt st, so that

$$\begin{array}{ccccc}
TX \otimes' I' & \xrightarrow{TX \otimes' e} & TX \otimes' JI & \xrightarrow{\mathrm{st}_{X,I}} & T(X \otimes I) \\
\downarrow{\scriptstyle \rho'_X} & & & & \downarrow{\scriptstyle \rho_X} \\
TX & & = & & TX
\end{array}$$

$$\begin{array}{ccccc}
(TX \otimes' JY) \otimes' JZ & \xrightarrow{\mathrm{st}_{X,Y} \otimes' JZ} & T(X \otimes Y) \otimes' JZ & \xrightarrow{\mathrm{st}_{X \otimes Y, Z}} & T((X \otimes Y) \otimes Z) \\
\downarrow{\scriptstyle \alpha'_{TX,TY,TZ}} & & & & \downarrow{\scriptstyle T\alpha_{X,Y,Z}} \\
TX \otimes' (JY \otimes' JZ) & \xrightarrow[TX \otimes' m_{Y,Z}]{} & TX \otimes' J(Y \otimes Z) & \xrightarrow[\mathrm{st}_{X,Y \otimes Z}]{} & T(X \otimes (Y \otimes Z))
\end{array}$$

$$\begin{array}{ccc} JX \otimes' JY & \xrightarrow{m_{X,Y}} & J(X \otimes Y) \\ {\scriptstyle \eta_X \otimes' JY}\downarrow & & \downarrow{\scriptstyle \eta_{X \otimes Y}} \\ TX \otimes' JY & \xrightarrow[\mathrm{st}_{X,Y}]{} & T(X \otimes Y) \end{array}$$

$$\begin{array}{ccc} JX \otimes' JY & \xrightarrow{m_{X,Y}} & J(X \otimes Y) \\ {\scriptstyle k \otimes' JY}\downarrow & & \downarrow{\scriptstyle \ell} \\ TX' \otimes' JY & \xrightarrow[\mathrm{st}_{X',Y}]{} & T(X' \otimes Y) \end{array} \quad \Rightarrow \quad \begin{array}{ccc} TX \otimes' JY & \xrightarrow{\mathrm{st}_{X,Y}} & T(X \otimes Y) \\ {\scriptstyle k^* \otimes' JY}\downarrow & & \downarrow{\scriptstyle \ell^*} \\ TX' \otimes' JY & \xrightarrow[\mathrm{st}_{X',Y}]{} & T(X' \otimes Y) \end{array}$$

Arrows

- Given a (small) monoidal category $(\mathbb{J}, I, \otimes)$.
- An arrow on $(\mathbb{J}, I, \otimes)$ is a weak arrow $(R, \mathrm{pure}, \lll)$ on $\mathbb{J}$ with, for any $X, Y, Z \in |\mathbb{J}|$, a map $\mathrm{first}_{X,Y,Z} \in R(X, Y) \to R(X \otimes Z, Y \otimes Z)$ satisfying
  - $\mathrm{pure}\,(\mathrm{id} \otimes f) \lll \mathrm{first}\, r = \mathrm{first}\, r \lll \mathrm{pure}\, \mathrm{id} \otimes f)$
  - $\mathrm{pure}\, \rho \lll \mathrm{first}\, r = r \lll \mathrm{pure}\, \rho$
  - $\mathrm{pure}\, \alpha \lll \mathrm{first}\,(\mathrm{first}\, r) = \mathrm{first} r \lll \mathrm{pure}\, \alpha$
  - $\mathrm{first}\,(\mathrm{pure}\, f) = \mathrm{pure}\,(f \otimes \mathrm{id})$
  - $\mathrm{first}\,(s \lll r) = \mathrm{first}\, s \lll \mathrm{first}\, r$
- $\mathrm{first}_{X,Y,Z}$ is natural in X, Y, dinatural in Z.

Arrows = strong relative monads on Yoneda

- Let $\mathbb{J}$ be small, take $\mathbb{C} =_{\mathrm{df}} [\mathbb{J}^{\mathrm{op}}, \mathbf{Set}]$, $J =_{\mathrm{df}} \mathbf{Y}$ (Yoneda on $\mathbb{J}$).
- A monoidal structure $(I, \otimes)$ on $\mathbb{J}$ induces one on $\mathbb{C}$ via
  - $I' Z =_{\mathrm{df}} \mathbb{J}(Z, I)$,
  - $(F \otimes' G)\, Z =_{\mathrm{df}} \int^{X,Y \in |\mathbb{J}|} \mathbb{J}(Z, X \otimes Y) \times (FX \times GY)$ (the Day convolution)
- $\mathbf{Y}$ becomes a monoidal functor.
- Consider a strong relative monad $(T, \eta, (-)^*, \mathrm{st})$.
- We have

  $$\begin{aligned} &(T\, X \otimes' \mathbf{Y}\ Y) Z \\ &\quad = \int^{X', Y' \in |\mathbb{J}|} \mathbb{J}(Z, X' \otimes Y') \times (T\, X\, X' \times \mathbb{J}(Y', Y)) \\ &\quad \cong \int^{X' \in |\mathbb{J}|} \mathbb{J}(Z, X' \otimes Y) \times T\, X\, X' \end{aligned}$$

- Hence

  $(\mathrm{st}_{X,Y})_Z \in \int^{X' \in |\mathbb{J}|} \mathbb{J}(Z, X' \otimes Y) \times T\, X\, X' \to T\, (X \otimes Y)\, Z$

  which is equivalent to having a map

  $\mathrm{first}_{X',X,Y} \in T\, X\, X' \to T\, (X \otimes Y)(X' \otimes Y)$

Arrows = strong monads in **Prof**

- Cf. Jacobs et al. (2006), Asada (2010)
- Arrows on a (small) category $\mathbb{J}$ are monoids in the category on the endoprofunctors on $\mathbb{J}$.
- Arrows are monads in the bicategory **Prof** of (small) categories and profunctors.

Strong monads vs strong relative monads

- If T is a strong monad on $(\mathbb{C}, I', \otimes')$, then $T^\flat =_{\mathrm{df}} T \cdot J$ is a strong relative monad on (J, e, m).
- If J is well-behaved, then
  if T is a strong relative monad on (J, e, m), then
  $T^\sharp =_{\mathrm{df}} \mathrm{Lan}_J\, T$ is a strong monad on $(C, I', \otimes')$.
- The adjunction

$$[\mathbb{C}, \mathbb{C}] \underset{\mathrm{Lan}_J}{\overset{- \cdot J}{\rightleftarrows}} [\mathbb{J}, \mathbb{C}] \qquad (\top)$$

  lifts to an adjunction

$$\mathbf{StrMnd}(\mathbb{C}, I', \otimes) \underset{(-)^\sharp}{\overset{(-)^\flat}{\rightleftarrows}} \mathbf{StrMnd}(J, e, m) \qquad (\top)$$

Conclusions

- Adding strength to relative monads is not difficult.
- Key idea: J must be a monoidal functor.
- Arrows become strong relative monads, are hence a natural structure.
  Hughes, Paterson got the axioms right without deriving arrows as an instance of something more general!

Future work

- Alternative descriptions of strong relative monads.
- Formalization in Agda.
- Arrow metalanguage (cf. Lindley, Wadler, Yallop 2010).