Dynamic Coalgebraic Modalities

Raul Andres Leal[1] & Helle Hvid Hansen[2]

[1]ILLC
Universiteit van Amsterdam

[2]Eindhoven University of Technology,
Centrum Wiskunde & Informatica, Amsterdam

CMCS 10, 2010 Cyprus

Outline

Outline

The long term aim

Plane: KLM 1951, requires landing authorisation at Schiphol Airport.

Tower: Schiphol tower, KLM 1951 descend to 1000mt. Expect heavy rain and strong tail wind during landing.

Plane: KLM 1951, Schiphol what should be landing procedure under these weather conditions?

Tower: Schiphol Tower, KLM 1951 after lowering the landing gear keep tail rudder still this will keep the aircraft stable.

The long term aim

State based system

Plane: KLM 1951, requires landing authorisation at Schiphol Airport.

Tower: Schiphol tower, KLM 1951 descend to 1000mt. Expect heavy rain and strong tail wind during landing.

Plane: KLM 1951, Schiphol what should be landing procedure under these weather conditions?

Tower: Schiphol Tower, KLM 1951 after lowering the landing gear keep tail rudder still this will keep the aircraft stable.

The long term aim

State based system

State of the system

Plane:	KLM 1951, requires landing authorisation at Schiphol Airport.
Tower:	Schiphol tower, KLM 1951 descend to 1000mt. Expect heavy rain and strong tail wind during landing.
Plane:	KLM 1951, Schiphol what should be landing procedure under these weather conditions?
Tower:	Schiphol Tower, KLM 1951 after lowering the landing gear keep tail rudder still this will keep the aircraft stable.

The long term aim

State based system

State of the system

Plane: KLM 1951, requires landing authorisation at Schiphol Airport.

Tower: Schiphol tower, KLM 1951 descend to 1000mt. Expect heavy rain and strong tail wind during landing.

Plane: KLM 1951, Schiphol what should be landing procedure under these weather conditions?

Tower: Schiphol Tower, KLM 1951 after lowering the landing gear keep tail rudder still this will keep the aircraft stable.

Programs

The long term aim

State based system

State of the system

Plane: KLM 1951, requires landing authorisation at Schiphol Airport.

Tower: Schiphol tower, KLM 1951 descend to 1000mt. Expect heavy rain and strong tail wind during landing.

Plane: KLM 1951, Schiphol what should be landing procedure under these weather conditions?

Tower: Schiphol Tower, KLM 1951 AFTER lowering the landing gear *keep tail rudder still* this will keep the aircraft stable.

Programs

The long term aim

State based system

State of the system

Plane: KLM 1951, requires landing authorisation at Schiphol Airport.

Tower: Schiphol tower, KLM 1951 descend to 1000mt. Expe… wind during landing.

Modal Statement

Plane: KLM 1951, Schiphol what should be landing procedure under these weather conditions?

Tower: Schiphol Tower, KLM 1951 AFTER lowering the landing gear *keep tail rudder still* this will keep the aircraft stable.

Programs

From Planes to Kripke Frames

Labelled transition systems.

These are coalgebras

$$S \to (\mathcal{P}S)^L$$

This yields to PDL, we reason about programs.

Monotone neighborhoods

Game/Coalition Frames

These are coalgebras

$$S \to (\mathcal{M}S)^L$$

This yields to Game Logic, we reason about strategic ability in 2-player games

From Planes to Kripke Frames

Labelled transition systems.

These are coalgebras

$$S \longrightarrow (\mathcal{P}S)^L$$

This yields to PDL, we reason about programs.

Game/Coalition Frames

These are coalgebras

$$S \longrightarrow (\mathcal{M}S)^L$$

This yields to Game Logic, we reason about strategic ability in 2-player games

From Planes to Kripke Frames

(free) algebra of regular expressions

Labelled transition systems.

These are coalgebras

$$S \longrightarrow (\mathcal{P}S)^L$$

This yields to PDL, we reason about programs.

Game/Coalition Frames

These are coalgebras

$$S \longrightarrow (\mathcal{M}S)^L$$

This yields to Game Logic, we reason about strategic ability in 2-player games

From Planes to Kripke Frames

(free) algebra of regular expressions

Labelled transition systems.

These are coalgebras

$$S \longrightarrow (\mathcal{P}S)^L$$

This yields to PDL, we reason about programs.

Game/Coalition Frames

These are coalgebras

$$S \longrightarrow (\mathcal{M}S)^L$$

This yields to Game Logic, we reason about strategic ability in 2-player games

(free) algebra of game expressions

Double perspective

Algebraic Perspective

$$\sigma : L \longrightarrow (GS)^S$$

Structure + Dynamics

Coalgebraic Perspective

$$\widehat{\sigma} : S \longrightarrow (GS)^L$$

Behavior + Modalities

Double perspective

Algebraic Perspective

$$\sigma : L \longrightarrow (GS)^S$$

Structure + Dynamics

Coalgebraic Perspective

$$\widehat{\sigma} : S \longrightarrow (GS)^L$$

Behavior + Modalities

Dynamic Modalities

Intuition

$s \Vdash \lambda^\alpha \varphi$ means "in state s, after α, φ holds".

PDL

$s \Vdash \Box^\alpha \varphi$ means "in state s, after transition α, φ holds".

Game Logic

$s \Vdash \Diamond^\alpha \varphi$ means "in state s, player 1 has a strategy in game α to bring about φ".

Dynamic Modalities

Intuition

$s \Vdash \lambda^{\alpha}\varphi$ means "in state s, after α, φ holds".

PDL

$s \Vdash \Box^{\alpha}\varphi$ means "in state s, after transition α, φ holds".

Game Logic

$s \Vdash \Diamond^{\alpha}\varphi$ means "in state s, player 1 has a strategy in game α to bring about φ".

Dynamic Modalities

Intuition

$s \Vdash \lambda^{\alpha}\varphi$ means "in state *s*, after α, φ holds".

PDL

$s \Vdash \Box^{\alpha}\varphi$ means "in state *s*, after transition α, φ holds".

Game Logic

$s \Vdash \Diamond^{\alpha}\varphi$ means "in state *s*, player 1 has a strategy in game α to bring about φ".

Dynamic Modalities

Intuition

$s \Vdash \lambda^\alpha \varphi$ means "in state s, after α, φ holds".

Labelling

Given a predicate lifting $\lambda : \mathcal{Q} \to \mathcal{Q}G$ and $\alpha \in L$, *the* α *labelling of* λ is a predicate lifting

$$\lambda^\alpha : \mathcal{Q} \to \mathcal{Q}G^L$$

given by

$$\lambda^\alpha(U) = \{\delta \in G(S)^L \mid \delta(\alpha) \in \lambda(U)\}$$

Outline

Describing composition of actions

PDL

Take $\Box = \lambda$, why does

$$\lambda^{\alpha;\beta}\varphi \iff \lambda^{\alpha}\lambda^{\beta}\varphi$$

hold?

Predicate transformers

Given $\sigma : S \to (GS)^L$ consider

$$([\alpha]^{\sigma}) \quad \mathcal{Q}S \xrightarrow{\lambda^{\alpha}_{S}} \mathcal{Q}(GS)^L \xrightarrow{\sigma^{-1}} \mathcal{Q}S$$

the equivalence above follows from

$$[\alpha;\beta]^{\sigma} = [\alpha]^{\sigma} \circ [\beta]^{\sigma}$$

Describing composition of actions

PDL

Take $\Box = \lambda$, why does

$$[\alpha;\beta]\varphi \iff [\alpha][\beta]\varphi$$

hold?

Predicate transformers

Given $\sigma : S \to (GS)^L$ consider

$$([\alpha]^\sigma) \quad \mathcal{Q}S \xrightarrow{\lambda^\alpha_S} \mathcal{Q}(GS)^L \xrightarrow{\sigma^{-1}} \mathcal{Q}S$$

the equivalence above follows from

$$[\alpha;\beta]^\sigma = [\alpha]^\sigma \circ [\beta]^\sigma$$

Describing composition of actions

PDL

Take $\Box = \lambda$, why does

$$\lambda^{\alpha;\beta}\varphi \iff \lambda^{\alpha}\lambda^{\beta}\varphi$$

hold?

Predicate transformers

Given $\sigma : S \to (GS)^L$ consider

$$([\alpha]^\sigma) \quad \mathcal{Q}S \xrightarrow{\lambda^\alpha_S} \mathcal{Q}(GS)^L \xrightarrow{\sigma^{-1}} \mathcal{Q}S$$

the equivalence above follows from

$$[\alpha;\beta]^\sigma = [\alpha]^\sigma \circ [\beta]^\sigma$$

Describing composition of actions

PDL

Take $\Box = \lambda$, why does

$$\lambda^{\alpha;\beta}\varphi \iff \lambda^{\alpha}\lambda^{\beta}\varphi$$

hold?

Predicate transformers

Given $\sigma : S \to (GS)^L$ consider

$$([\alpha]^\sigma) \quad \mathcal{Q}S \xrightarrow{\lambda^\alpha_S} \mathcal{Q}(GS)^L \xrightarrow{\sigma^{-1}} \mathcal{Q}S$$

the equivalence above follows from

$$[\alpha;\beta]^\sigma = [\alpha]^\sigma \circ [\beta]^\sigma$$

Monads for composition

Theorem

Let λ be a predicate lifting. If sequential composition is interpreted as Kleisli composition, then

$$\lambda^{\alpha;\beta}\varphi \iff \lambda^{\alpha}\lambda^{\beta}\varphi$$

holds if one of the following conditions hold . . .

- *the transpose $\widehat{\lambda} : G \to QQ$ is a monad morphism.*
- *The algebra $Y(\lambda) : G2 \to 2$ is a G-algebra (monads).*

Monads for composition

Theorem

Let λ be a predicate lifting. If sequential composition is interpreted as Kleisli composition, then

$$[\alpha; \beta]^\sigma = [\alpha]^\sigma \circ [\beta]^\sigma$$

holds if one of the following conditions hold ...

- *the transpose $\widehat{\lambda} : G \longrightarrow QQ$ is a monad morphism.*
- *The algebra $Y(\lambda) : G2 \longrightarrow 2$ is a G-algebra (monads).*

Monads for composition

Theorem

Let λ be a predicate lifting. If sequential composition is interpreted as Kleisli composition, then

$$\lambda^{\alpha;\beta}\varphi \iff \lambda^{\alpha}\lambda^{\beta}\varphi$$

holds if one of the following conditions hold ...

- *the transpose $\widehat{\lambda} : G \to QQ$ is a monad morphism.*
- *The algebra $Y(\lambda) : G2 \to 2$ is a G-algebra (monads).*

Monads for composition

Theorem

Let λ be a predicate lifting. If sequential composition is interpreted as Kleisli composition, then

$$\lambda^{\alpha;\beta}\varphi \iff \lambda^{\alpha}\lambda^{\beta}\varphi$$

holds if one of the following conditions hold ...

- *the transpose $\widehat{\lambda} : G \to \mathcal{Q}\mathcal{Q}$ is a monad morphism.*
- *The algebra $Y(\lambda) : G2 \to 2$ is a G-algebra (monads).*

Monads for composition

Theorem

Let λ be a predicate lifting. If sequential composition is interpreted as Kleisli composition, then

$$\lambda^{\alpha;\beta}\varphi \iff \lambda^{\alpha}\lambda^{\beta}\varphi$$

holds if one of the following conditions hold . . .

- *the transpose $\widehat{\lambda} : G \to QQ$ is a monad morphism.*
- *The algebra $Y(\lambda) : G2 \to 2$ is a G-algebra (monads).*

Yoneda

Monads for composition

Theorem

Let λ be a predicate lifting. If sequential composition is interpreted as Kleisli composition, then

$$\lambda^{\alpha;\beta}\varphi \iff \lambda^{\alpha}\lambda^{\beta}\varphi$$

holds if one of the following conditions hold ...

- *the transpose $\widehat{\lambda} : G \to \mathcal{Q}\mathcal{Q}$ is a monad morphism.*
- *The algebra $Y(\lambda) : G2 \to 2$ is a G-algebra (monads).*

Monads for composition

Theorem

Let λ be a predicate lifting. If sequential composition is interpreted as Kleisli composition, then

$$\lambda^{\alpha;\beta}\varphi \iff \lambda^{\alpha}\lambda^{\beta}\varphi$$

holds if one of the following conditions hold ...

- *the transpose $\widehat{\lambda} : G \to \mathcal{Q}\mathcal{Q}$ is a monad morphism.*
- *The algebra $Y(\lambda) : G2 \to 2$ is a G-algebra (monads).*

Outline

Towards Planes; more complex operations

Other Operations

How do we obtain axioms like

$$\lambda^{\alpha \cup \beta} \varphi \iff \lambda^{\alpha} \varphi \wedge \lambda^{\beta} \varphi?$$

Towards Planes; more complex operations

Other Operations

How do we obtain axioms like

$$\lambda^{\alpha\cup\beta}\varphi \iff \lambda^{\alpha}\varphi \wedge \lambda^{\beta}\varphi?$$

Answer 1: There is an enriched functor

$$\widehat{\lambda} \circ - : \mathcal{Kl}(G) \to \mathcal{Kl}(\mathcal{Q}\mathcal{Q})$$

Towards Planes; more complex operations

Other Operations

How do we obtain axioms like

$$\lambda^{\alpha \cup \beta}\varphi \iff \lambda^{\alpha}\varphi \wedge \lambda^{\beta}\varphi?$$

Answer 2: $\widehat{\lambda}$ is a homomorphism, i.e. a diagram like

$$\begin{array}{ccc} TG & \xrightarrow{T(\widehat{\lambda})} & TQQ \\ \downarrow & & \downarrow \\ G & \xrightarrow[\widehat{\lambda}]{} & QQ \end{array}$$

commutes.

Towards Planes; more complex operations

Other Operations

How do we obtain axioms like

$$\lambda^{\alpha\cup\beta}\varphi \iff \lambda^{\alpha}\varphi \wedge \lambda^{\beta}\varphi ?$$

Answer 2: in PDL... $\widehat{\Box}$ is a homomorphism.

$$\begin{array}{ccc} T\mathcal{P} & \xrightarrow{T(\widehat{\Box})} & TQQ \\ \downarrow & & \downarrow \\ \mathcal{P} & \xrightarrow[\widehat{\Box}]{} & QQ \end{array}$$

commutes.

Towards Planes; more complex operations

Other Operations

How do we obtain axioms like

$$\lambda^{\alpha\cup\beta}\varphi \iff \lambda^{\alpha}\varphi \wedge \lambda^{\beta}\varphi?$$

Answer 2: in Game Logic... $\widehat{\Box}$ is NOT a homomorphism.

$$\begin{array}{ccc} T\mathcal{M} & \xrightarrow{T(\widehat{\Box})} & TQQ \\ \downarrow & & \downarrow \\ \mathcal{M} & \xrightarrow[\widehat{\Box}]{} & QQ \end{array}$$

commutes.

Other issues

Input/output

We do not understand how to deal with input/output (functors that are not monads)

$$(\mathit{Java}) \quad F(S) := (1 + S \times B + S \times E)^A$$

Idea: Use

$$J(B) := (1 + S \times B + S \times E)^S$$

which is a monad.

Problem: Actions are subject to typing conditions.

Other issues

Input/output

We do not understand how to deal with input/output (functors that are not monads)

$$(Java) \quad F(S) := (1 + S \times B + S \times E)^A$$

Idea: Use

$$J(B) := (1 + S \times B + S \times E)^S$$

which is a monad.

Problem: Actions are subject to typing conditions.

Other issues

Input/output

We do not understand how to deal with input/output (functors that are not monads)

$$(\mathit{Java}) \quad F(S) := (1 + S \times B + S \times E)^A$$

Idea: Use

$$J(B) := (1 + S \times B + S \times E)^S$$

which is a monad.

Problem: Actions are subject to typing conditions.

Other issues

Definability

We can now define operations on the label even if they make “no sense” for the coalgebra; e.g.

$$\lambda^{\alpha \cup \beta} = \lambda^{\alpha} \cup \lambda^{\beta}.$$

When are those definable and what do they express is unclear to us.

The End!!!

- We understand how to label modalities.
- We can explain the axiom of sequential composition.
- We can explain axioms for algebraic operations.
- We can not see any bialgebra.
- The general picture is still unclear.
- The Test modality is still evasive.
- Input/output should be worked out.

The End!!!

- We understand how to label modalities.
- We can explain the axiom of sequential composition.
- We can explain axioms for algebraic operations.
- We can not see any bialgebra.
- The general picture is still unclear.
- The Test modality is still evasive.
- Input/output should be worked out.