Technische Universität Braunschweig

Institut für Theoretische Informatik

Recursive Program Schemes with Effects

Daniel Schwencke, 28th March 2010

Outline

RPSs

Idea: define new operations using given operations and recursion

Definition (RPS without effects, classical)

- disjoint finite sets F – given operation symbols
  Φ – new operation symbols
  X – variables
- $\phi(x_1, \dots, x_n) \approx t^\phi(x_1, \dots, x_n)$ for all $\phi \in \Phi_n$, t^ϕ term in $F \cup \Phi$

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Example ([Milius Moss 06])

$$\phi(x) \approx f(x, \phi(gx))$$
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Generalising category-theoretic approach in
[Ghani Lüth de Marchi 03, Milius Moss 06]

ND-RPSs

Idea: add non-deterministic choice on rhs of formal equations

- special binary operation symbol $or \notin F \cup \Phi$
- terms t^{ϕ} in $F \cup \Phi \cup \{or\}$
- see [Arnold Nivat 77]

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More generally: RPSs with effects

- partiality
- non-determinism
- probabilism

A Starting Point

Assumptions

- (M, η^M, μ^M) *monad on* **Set**
- H, V *finitary* **Set**-*functors*
- *distributive laws* $\lambda : HM \to MH$ *and* $\nu : VM \to MV$

$\Rightarrow$ induced distributive law $\rho : (H + V)M \to M(H + V)$

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Meaning:

- M – effect, e. g. $\_ + 1$, $\mathcal{P}$, $\mathcal{D}$
- H, V – "signatures" of given/new operations
- λ, ν, ρ – extension of operations to parameters with effects

A First Lemma

Notation:

- (F^G, η^G, μ^G) free monad on G
- universal natural transformation $\kappa^G : G \to F^G$
- T monad, $\sigma : G \to T$. Then
  $\sigma^\# : F^G \to T$ unique monad morphism such that $\sigma^\# \cdot \kappa^G = \sigma$

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Lemma

If G has free algebras, every distributive law $\delta : GM \to MG$ induces a distributive law $\delta' : F^G M \to MF^G$.

$\Rightarrow$ composite monad $(MF^G, \eta^M F^G \cdot \eta^G, (\mu^M * \mu^G) \cdot M\delta' F^G)$

RPSs with Effects

Definition

- *M*-RPS $e : V \to MF^{H+V}$
- guarded if $e \equiv V \xrightarrow{e_0} M(HF^{H+V} + \mathrm{Id}) \xrightarrow{\cdots} MF^{H+V}$
- (uninterpreted) solution of e $e^\dagger : V \to MF^H$ such that $e^\dagger = \mu^M F^H \cdot M[\eta^M F^H \cdot \eta^H, e^\dagger]^\# \cdot e$

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Example

For $pow(x) \approx x\ or\ (x \cdot pow(x))$ take $M = \mathcal{P}$, $V = \mathrm{Id}$, $H = \mathrm{Id}^2$

- $e_X(x) = \{x, x \cdot pow(x)\}$
- guarded since $x \in \mathrm{Id}(X)$ and $x \cdot pow(x) \in HF^{H+V}X$
- $e_X^\dagger(x) = \{x, x \cdot x, x \cdot (x \cdot x), x \cdot (x \cdot (x \cdot x)), \dots\}$ is a solution

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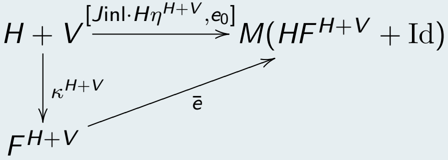
- a monad $\mathcal{H}F^{H+V}$ with distributive law over M.

Second Order Substitution with Effects

Definition

For a guarded M-RPS e let $\bar{e}$ be the unique monad morphism such that the diagram commutes:



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$$\begin{array}{ccc} H + V & \xrightarrow{[\mathsf{Jinl} \cdot H\eta^{H+V}, e_0]} & M(HF^{H+V} + \mathrm{Id}) \\ \big\downarrow \kappa^{H+V} & \nearrow \bar{e} & \\ F^{H+V} & & \end{array}$$

Remarks

- $\bar{e}$ *performs second order substitution with effect handling*
- $\bar{e}$ *is an* $\bar{\mathcal{H}}$*-coalgebra*

Sufficient Conditions for a Solution

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- $[\phi^H, \eta^H] : \mathcal{H} F^H \to F^H$ is initial $\mathcal{H}$-algebra.
- If
  1. $J[\phi^H, \eta^H]^{-1} : F^H \to \bar{\mathcal{H}} F^H$ is final $\bar{\mathcal{H}}$-coalgebra and
  2. the unique $\bar{\mathcal{H}}$-coalgebra homomorphism $h : F^{H+V} \to MF^H$ between $\bar{e}$ and $J[\phi^H, \eta^H]^{-1}$ is a monad morphism

  then $h \cdot \kappa^{H+V} \cdot \mathsf{inr} : V \to MF^H$ is a solution of e.

A Result for CPO-enriched $\mathbf{Set}_M$

Assumptions

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Theorem

Under the above assumptions, every guarded M-RPS has a solution.

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1. $J[\phi^H, \eta^H]^{-1}$ final $\bar{\mathcal{H}}$-coalgebra: use techniques of [Hasuo Jacobs Sokolova 07]

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Examples ([Milius Palm S 09])

Monads $\_ + 1$, $\mathcal{P}$ or $\mathcal{D}$ with analytic H and canonical λ

Future Work

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2. generalise M-RPS-definition to allow CIMs
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2. generalise M-RPS-definition to allow CIMs
   - [Arnold Nivat 77]-setting category-theoretic
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3. interpreted solutions using [Milius Palm S 09]

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Thank you. . .

. . . for your attention!

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