Coinduction in concurrent timed systems

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Outline

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Coalgebra and automata theory

- Labelled transition systems (incl. timed) are coalgebras
- Various automata are coalgebras of suitable set functors
- Weighted automata (automata with multiplicities) are coalgebras
- Deterministic automata have simple final coalgebras: e.g. languages, formal power series (Moore automata)
- 2 ways of coding concurrency using weighted automata : nondeterminism (heap automata) and synchronous composition (like timed automata)
- Classes of timed automata (product interval automata) and corresponding classes of Petri nets
- Behaviors of synchronous compositions

Deterministic K-weighted automata as coalgebras

- Mealy automata (inputs in A, outputs in K) are coalgebras (S, t), S set of states, $t : S \to (K \times S)^A$ transition function.
- A partial MA is (S, t), where $t : S \to (1 + (K \times S))^A$ with $1 = \{\emptyset\}$.
- Partial Mealy automata are deterministic K-weighted automata with all states final
- A deterministic K-weighted automaton is viewed as partial Mealy automaton (S, t) above.
- Examples of multiplicity semirings :
    - $K = \mathbb{R}_{min} = (\mathbb{R} \cup \{\infty\}, min, +, \infty, 0)$ ... (min,+)-automata (price)
    - $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$ ... (max,+)-automata (time)
    - $K = \mathcal{I}_{max}^{max} = (\mathbb{R}_{\max} \times \mathbb{R}_{\max} \cup (-\infty, -\infty), \oplus, \otimes, (-\infty, -\infty), (0, 0))$ ... interval automaton
    - $(R^+, +, \times, 0, 1)$... stochastic automata (probability semiring)

Outline

Stream coalgebra

Streams are infinite sequences over a set, e.g. a semiring $K = (K, \oplus, \otimes, 0, 1)$.

$(K^\omega, \langle head, tail \rangle)$ is the *final coalgebra* of $F(S) = K \times S$.
Definition. For $s = (s(0), s(1), s(2), s(3), \dots) \in K^\omega$:
$head(s) = s(0)$ and $tail(s) = s' = (s(1), s(2), s(3), \dots)$.

Other notation:

$[r] = (r, 0, 0, \dots) \dots$ constant stream for $r \in K$.

$X = (0, 1, 0, \dots) \dots$ important to describe any stream

Final Mealy automaton

- Behaviors of Mealy automata are causal stream functions $f : A^\omega \to K^\omega$. $f :\ A^\omega \to K^\omega$ is *causal* if $\forall n \in \mathbb{N}$, $\sigma, \tau \in A^\infty$: $\forall i :\ i \leq n$: $\sigma(i) = \tau(i)$ then $f(\sigma)(n) = f(\tau)(n)$.
- Stream derivatives: $\omega = (\omega_0, \omega_1, \dots) \in K^\omega,\ \omega \to \omega' = (\omega_1, \dots)$.
- Stream functions form final coalgebra of Mealy automata with $t(f) = \langle f[a], f_a\rangle$ $f[a] = f(a : \sigma)(0)$ and $f_a(\sigma) = f(a : \sigma)'$
- For partial Mealy automata consider $f :\ A^\omega \to (1+K)^\omega$ f is *consistent* if $\sigma \in A^\omega$: $f(\sigma)(k) = \emptyset$ then $f(\sigma)(n) = \emptyset$ for any $n > k$.
- $\mathcal{F} = (\mathcal{F}, t_{\mathcal{F}})$ is the final coalgebra of partial Mealy automata: $\mathcal{F} = \{f :\ A^\omega \to (1+K)^\omega \ | f \text{ is causal and consistent}\}$.

$$t_{\mathcal{F}}(f)(a) = \begin{cases} \langle f[a], f_a\rangle & \text{if } f[a] \neq \emptyset \in 1, \\ \emptyset & \text{otherwise,} \end{cases}$$

Equivalent presentation of behaviors

- $s_0 \stackrel{\sigma(0)|k_0}{\rightarrow} s_1 \stackrel{\sigma(1)|k_1}{\rightarrow} s_2 \cdots \stackrel{\sigma(n)|k_n}{\rightarrow} s_{n+1}$. We define

$$l(s_0)(\sigma)(n) = k_n.$$

- $A^\infty = A^\omega \cup A^+$, where $A^+ = A^* \setminus \{\lambda\}$
- $\mathcal{F}$ is isomorphic to functions between finite and infinite sequences!

$$\mathcal{F}_\infty = \{f : A^\infty \to K^\infty | f \text{ length preserving, causal, } dom(f) \text{ prefix-closed}\}$$

- $f[a] = f(a)(0)$ whenever f is defined for $a \in A$.
- $f_a : A^\infty \to (1 + K)^\infty$ given by $f_a(s) = f(a : s)'$
-

$$t_{\mathcal{F}_\infty}(f)(a) = \begin{cases} \langle f[a], f_a \rangle & \text{if } f[a] \text{ is defined} \\ \text{undefined} & \text{otherwise,} \end{cases}$$

Fundamental theorem of stream functionals

Fundamental theorem of stream calculus:

$$\sigma = \sigma(0) \oplus X\sigma'(0) \oplus X^2\sigma''(0) \oplus \dots$$

has its stream functional counterpart:

Theorem. For any $f \in \mathcal{F}$ and $\sigma = (\sigma(0), \sigma(1), \dots, \sigma(k), \dots) \in A^\omega$ we have:

$$f(\sigma) = f(\sigma)(0) \oplus X f_{\sigma(0)}(\sigma')(0) \oplus \dots X^k f_{\sigma(0)\dots,\sigma(k-1)}(\omega^{(k)})(0) \oplus \dots$$

or equivalently,

$$f(\sigma) = f[\sigma(0)] \oplus X f_{\sigma(0)}[\sigma(1)] \oplus \dots X^k f_{\sigma(0)\dots,\sigma(k-1)}[\sigma(k)] \oplus \dots$$

Proposition.

1. For any $f \in \mathcal{F}_\infty$, $\omega \in A^\infty$, and $a \in A$: $f(a) : f_a(\omega) = f(a\omega)$.
2. More generally, for any $u \in A^+$ and $\omega \in A^\infty$: $f(u) : f_u(\omega) = f(u\omega)$.

Properties of stream functionals

Initial output is a particular partial stream functional defined by

$$f^{\infty}[a](\sigma) = \begin{cases} f[a] & \text{if } \sigma = a, \\ \text{undefined} & \text{otherwise: } \sigma \neq a, \end{cases}$$

Definition. For $f, g \in \mathcal{F}_{\infty}$, $\sigma = (\sigma(0) : \sigma') \in A^{\infty}$, and $a \in A$ we define

$$(f^{\infty}[a] \odot g)(\sigma(0) : \sigma') = \begin{cases} f(\sigma(0)) : g(\sigma') & \text{if } a = \sigma(0) \in dom(f), \\ \text{undefined} & \text{otherwise,} \end{cases}$$

Theorem 1. For any $f \in \mathcal{F}_{\infty}$ we have: $f = \bigoplus_{a \in A} f^{\infty}[a] \odot f_a$.

Theorem 2. For any $f \in \mathcal{F}_{\infty}$ and $a \in A$: $(f^{\infty}[a] \odot f)_a = f$

Outline

(max,+)- automata

- (max,+) automata are $G = (Q, \alpha, t, \beta)$, where Q is a finite set of states, $\alpha : Q \to \mathbb{R}_{max}$, $t : Q \times A \times Q \to \mathbb{R}_{max}$, and $\beta : Q \to \mathbb{R}_{max}$, called initial, transition, and final delays.
- Also: $G = (Q, A, q_0, Q_m, t)$, where
  - A set of discrete events,
  - q_0 initial state, Q_m subset of final or marked states,
  - $t : Q \times A \times Q \to \mathbb{R}_{max}$ transition function

Meaning: output value $t(q, a, q') \in \mathbb{R}_{max}$ corresponds to the duration of $a-$transition from q to q' and

$t(q, a, q') = \varepsilon$ if there is no transition from q to q' labeled by a.

Algebraic behaviors of (max,+)- automata

Formal power series with variables in A and coefficients in $\mathbb{R}_{max}$.

$\mathbb{R}_{max}(A)$ isomorphic to $\{\omega : A^* \to \mathbb{R}_{max}\}$.

Behavior of $G = \langle Q, A, q_0, Q_m, t\rangle$ for $w = a_1 \dots a_n \in A^*$:

$$l(G)(w) = \max_{q_1,\dots,q_n \in Q:\ q_n \in Q_m} (t(q_o, a_1, q_1) + t(q_1, a_2, q_2) + \cdots + t(q_{n-1}, a_n, q_n)).$$

$l(G)(w)$ is the longest path corresponding to label w from the initial state to a final state.

Using the matrix formalism:

$$l(G)(w) = \alpha \otimes t(w) \otimes \beta,$$

typically $\alpha = (e, \varepsilon, \dots, \varepsilon)$ and similarly for β

Unambiguous and deterministic (max,+)- automata

- (max,+) automata are seemingly simple, still powerful model, cf. 1- safe (timed) Petri nets!
- A K-automaton is unambiguous if, for every word w, there is at most one successful path labeled by w.
- Unambiguous series: ∃unambiguous automaton recognizing it.
- Lombardy and Mairesse: unambiguous series are intersection of (max,+) and (min,+)-rational series
- Beyond unambiguous series equality and inequality is undecidable and no rational controllers exist!
- Decidable classes of timed automata : one clock timed automata (e.g. interval automata)
  and their synchronous products called product interval automata compositions

(max,+)- automata coalgebraically

det. (max,+)- automata $\quad S = (S, t),\ t : S \to (1 + (\mathbb{R}_{\max} \times S))^A$

A *homomorphism* between $S = (S, t)$ and $S' = (S', t')$ is $f : S \to S'$ s.t. $\forall s \in S$ and $\forall a \in A$: if $s \stackrel{a|b}{\to} s'$ then $f(s) \stackrel{a|b}{\to} f(s')$, i.e.:

$$\begin{array}{ccc}
(1 + (\mathbb{R}_{\max} \times S))^A & \stackrel{t}{\longleftarrow} & S \\
\Big\downarrow F(f) & & \Big\downarrow f \\
(1 + (\mathbb{R}_{\max} \times S'))^A & \stackrel{t'}{\longleftarrow} & S'
\end{array}$$

A *bisimulation* between $S = (S, t)$ and $S' = (S', t')$ is $R \subseteq S \times S'$ s.t. $\forall s \in S$ and $\forall s' \in S'$: if $\langle s, s' \rangle \in R$ then

(i) $\forall a \in A$: $s \stackrel{a}{\not\to}$ iff $s' \stackrel{a}{\not\to}$

(ii) $\forall a \in A: \ s \stackrel{a|b}{\to} q \ \Rightarrow \ s' \stackrel{a|b'}{\to} q'$ s. t. $\langle q, q' \rangle \in R, b = b'$, and

(iii) $\forall a \in A: \ s' \stackrel{a|b'}{\to} q' \ \Rightarrow \ s \stackrel{a|b}{\to} q$ such that $\langle q, q' \rangle \in R$, and $b = b'$.

Behaviors of (max,+)-automata

Algebraic behaviors: formal power series

Coalgebraic behaviors: stream functions from $\mathcal{F}$.

$\mathcal{F} = \{f : A^\infty \to \mathbb{R}_{\max}^\infty | f$ length preserving, causal, $dom(f)$ prefix-closed$\}$.

Similar to timed languages: $L_t \subseteq (A \times \mathbb{R})^\infty$, but tailored to one clock timed automata!

Timed languages give the cumulated execution time of a sequence!

Stream functions give the duration of events in the sequence!

Outline

Synchronous extensions of (max,+)-automata

Major problem : (max,+)-automata as a class of timed automata are **not closed** under synchronous composition!

Our Solution: using extended multi-event alphabets
Let G_1 and G_2 be (max,+) automata over local alphabets A_1 and A_2. Associated natural projections are denoted by: $P_1 : (A_1 \cup A_2)^* \to A_1^*$ et $P_2 : (A_1 \cup A_2)^* \to A_2^*$. Boolean morphism matrices are needed:

$$[B\mu(a)]_{ij} = \begin{cases} e, & \text{if } [\mu(a)]_{ij} \neq \varepsilon \\ \varepsilon, & \text{else} \end{cases}$$

To alleviate notation $B(a)$ instead of $B\mu(a)$.
This can be extended to words of A^* by:

$$B(a_1 \dots a_n) = B(a_1) \dots B(a_n).$$

Tensor linear algebra

If A is a matrix of dimension $m \times n$ and B a matrix of dimension $p \times q$ over a dioid, their tensor (Kronecker) product $A \otimes^t B$ is the matrix of dimension $mp \times nq$:

$$A \otimes^t B = \begin{bmatrix} a_{11}B & \cdots & a_{11}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

In particular, for square matrices $A = (a_{ij})_{i,j=1}^{n}$ and $B = (b_{kl})_{k,l=1}^{m}$, $C = A \otimes^t B$ is a matrix of dimension $n.m \times n.m$ with

$$C_{ik,jl} = a_{ij} \otimes^t b_{kl}.$$

Synchronous composition using extended alphabets

Definition. (Synchronous product)
Synchronous product of (max,+) automata
$G_1 = (Q_1, A_1, \alpha_1, \mu_1, \beta_1)$ and $G_2 = (Q_2, A_2, \alpha_2, \mu_2, \beta_2)$, is (max,+) automaton defined over alphabet

$$\mathcal{A} = (A_1 \cap A_2) \cup (A_1 \setminus A_2)^* \times (A_2 \setminus A_1)^*$$

by

$$G_1 \| G_2 = \mathcal{G} = (Q_1 \times Q_2, \mathcal{A}, \alpha, \mu, \beta)$$

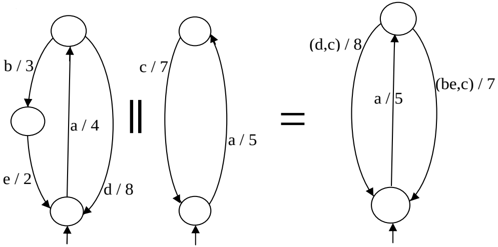
with $Q_1 \times Q_2$ state set, $\mathcal{A}$ event set, $\alpha = \alpha_1 \otimes^t \alpha_2$ initial delays, $\mu : \mathcal{A}^* \to \mathbb{R}_{max}^{|Q| \times |Q|}$ morphism matrix and $\beta = \beta_1 \otimes^t \beta_2$ final delays.
Morphism matrix :

$$\mu(v) = \begin{cases} \mu_1(v) \otimes^t B_2(v) \oplus B_1(v) \otimes^t \mu_2(v), & \text{if } v = a \in A_1 \cap A_2 \\ \mu_1(P_1(v)) \otimes^t B_2(P_2(v)) \oplus B_1(P_1(v)) \otimes^t \mu_2(P_2(v)), & \text{if } v = (P_1(v), P_2(v)) \end{cases}$$

Illustration of the synchronous product

$$\mathbf{G_1} \| \mathbf{G_2} = \mathcal{G} = (\mathbf{Q_1} \times \mathbf{Q_2}, \mathcal{A}, \alpha, \mu, \beta), \text{where}$$

$$\mathcal{A} = \{a, (be, c), (d, c)\} \subseteq (A_1 \cap A_2) \cup (A \setminus (A_1 \cap A_2))^*,$$



$\alpha = \alpha_1 \otimes^t \alpha_2$, $\beta = \beta_1 \otimes^t \beta_2$, and

$$\nu(v) = \begin{cases} \mu_1(a) \otimes^t B_2(a) \oplus B_1(a) \otimes^t \mu_2(a), & \text{if } v = a \in A_1 \cap A_2 \\ \mu_1(be) \otimes^t B_2(c) \oplus B_1(be) \otimes^t \mu_2(c), & \text{if } v = (be, c) \\ \mu_1(d) \otimes^t B_2(c) \oplus B_1(d) \otimes^t \mu_2(c), & \text{if } v = (d, c) \end{cases}$$

Induced behavior

- Behaviors of $G_1 \| G_2$ are formal power series of $\mathbb{R}_{\max}(\mathcal{A})$. From practical viewpoint (performance analysis, control) $I(G_1 \| G_2)(w)$ for $w \in A^*$ are more interesting (durations of tasks)
- Any $w \in A^*$ admits decomposition $w = v_0 a_1 v_1 \dots a_n v_n$, with $a_i \in A_1 \cap A_2,\ i = 1, \dots, n$ shared events and $v_i \in (A \setminus (A_1 \cap A_2))^*,\ i = 0, \dots, n$ private sequences.
- The local tasks of G_1 and G_2 corresponding to v_i are given by $P_1(v_i)$ et $P_2(v_i)$, resp.
- Any word from A^* can be seen as an element of $\mathcal{A}^*$, namely

$$w = P_1(v_0) \times P_2(v_0).a_1 P_1(v_1) \times P_2(v_1) \dots a_n P_1(v_n) \times P_2(v_n)$$

- Morphism μ induces the matrix mapping $\nu : A^* \to \mathbb{R}_{\max}$:

$$\nu(w) = \mu(P_1(v_0) \times P_2(v_0))\mu(a_1)\mu(P_1(v_1) \times P_2(v_1)) \dots \mu(a_n)\mu(P_1(v_n) \times P_2(v_n)).$$

Induced behavior continued

Definition. Induced behavior of $G_1 \| G_2$ is given by:

$$I(G_1 \| G_2)(w) = \alpha \nu(w) \beta.$$

Notation: $Z = \{\nu, B\}$ with complement $\bar{\nu} = B$ and $\bar{B} = \nu$.
Extension to words: $m = m^1 \dots m^k$, $\bar{m} = \bar{m}^1 \dots \bar{m}^k$.
Theorem. Induced behavior of $G_1 \| G_2$ for $w = v_0 a_1 v_1 \dots a_n v_n \in A^*$ is :

$$I(G_1 \| G_2)(w) = \bigoplus_{m \in Z^{2n+1}} \alpha_1 m_1(P_1(w)) \beta_1 \otimes \alpha_2 \bar{m}_2(P_2(w)) \otimes \beta_2.$$

Special case n=0.

$$l_1 \| l_2 = l_1(P_1(w)) \otimes supp(l_2)(P_2(w)) \oplus l_2(P_2(w)) \otimes supp(l_1)(P_1(w)),$$

because $supp(l_i)(P_i(w)) = \alpha_i B_i(P_i(w)) \beta_i$ for $i = 1, 2$.
Hint for better understanding.
$L_1 \| L_2 = P_1^{-1}(L_1) \cap P_2^{-1}(L_2)$, in terms of Boolean series:

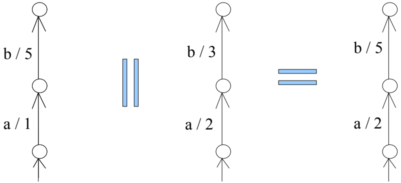
$$L_1 \| L_2(w) = L_1(P_1 w) \otimes L_2(P_2 w).$$

Synchronous product of series: example

Example shows that there is no simple formula for $l_1 \| l_2(w)$.

Here, $l_1 = 1a \oplus 6ab$, $l_2 = 2a \oplus 5ab$, and $l_1 \| l_2 = 2a \oplus 7ab$.

Linear representations of l_i are needed!



Synchronous product defined by coinduction

For $l_i \in \mathcal{F}$ over A_i and $v_i = a_1 \dots a_k \in A_i^+$ we define for $i = 1, 2$:

$$(l_i)[v_i] = (l_i)[a_1] \otimes (l_i)_{a_1}[a_2] \otimes \cdots \otimes (l_i)_{a_1 \dots a_{k-1}}[a_k].$$

Definition. for $l_1, l_2 \in \mathcal{F}$ and $\forall v \in \mathcal{A}$:

$$\begin{aligned} (l_1 \| l_2)_v &= (l_1)_{P_1(v)} \| (l_2)_{P_2(v)} \text{ and} \\ (l_1 \| l_2)[v] &= l_1[P_1(v)] \otimes Bl_2[P_2(v)] \oplus Bl_1[P_1(v)] \otimes l_2[P_2(v)]. \end{aligned}$$

Special case with full synchronization ($A_1 = A_2$): no need for using extended alphabet, in fact $\mathcal{A} = A_1 = A_2$.
For $v = a \in \mathcal{A}$ we have in fact $P_1(v) = P_2(v) = a$. Hence,

$$(l_1 \| l_2)_a = (l_1)_a \| (l_2)_a$$

and $(l_1 \| l_2)[a] = l_1[a] \otimes Bl_2[a] \oplus Bl_1[a] \otimes l_2[a]$.

Synchronous product continued

Equivalent expression for first input:

$$(l_1 \| l_2)[v] = \begin{cases} max(l_1[P_1(v)], l_2[P_2(v)]) & \text{if } l_i[P_i(v)] \neq \varepsilon \text{ for } i = 1,2 \\ \varepsilon & \text{else, i.e. } \exists i = 1,2 : l_i[P_i(v)] = \varepsilon \end{cases}$$

Hint for understanding:

for partial languages $L_1 = (L_1^1, L_1^2)$, $L_2 = (L_2^1, L_2^2)$, and $w \in A^*$ we have in fact

$$(L_1 \| L_2)_w = (L_1)_{P_1(w)} \| (L_2)_{P_2(w)}.$$

Behavior of synchronous product: example

$$(l_1 \| l_2)(a) = (l_1 \| l_2)(a)(0) = 5 = (l_1 \| l_2)[a]$$

$$(l_1 \| l_2)(a(d,c)) = (l_1 \| l_2)(a(d,c))(0) \oplus X(l_1 \| l_2)_a(d,c).$$

Formulas for derivative and first output function yield:

$$(l_1 \| l_2)_a(dc) = ((l_1)_a \| (l_2)_a)(dc) = ((l_1)_a \| (l_2)_a)(dc)(0) = ((l_1)_a \| (l_2)_a)[dc]$$

$$= (l_1)_a[d] \otimes B(l_2)_a[c] \oplus B(l_1)_a[d] \otimes (l_2)_a[c] = (l_1)(ad)(1) \otimes B(l_2)(ac)(1) \oplus$$

$$B(l_1)(ad)(1) \otimes (l_2)(ac)(1) = 8 \otimes 0 \oplus 0 \times 7 = 8.$$

Note that $(l_1)_a[d] = (l_1)_a(d)(0) = (l_1)(a:d)'(0) = (l_1)(ad)(1)$.
Similarly,
$(l_1 \| l_2)(a(be,c)) = (l_1 \| l_2)(a(be,c))(0) \oplus (l_1 \| l_2)_a((be,c))$, where

$$(l_1 \| l_2)_a(be,c) \quad = \quad \cdots = (l_1)(a(be))(1) \otimes B(l_2)(ac)(1) \oplus$$

$$B(l_1)(a(be))(1) \otimes (l_2)(ac)(1) = 5 \otimes 0 \oplus 0 \times 7 = 7.$$

Induced behavior example algebraically

- Induced behaviors translate series from $\mathbb{R}_{\max}(\mathcal{A})$, i.e. over $\mathcal{A}$ into standard series from $\mathbb{R}_{\max}(A)$ and correspond to duration of distributed tasks
- In accordance with Proposition we obtain for $w = abec$ the induced behavior

$$I(abec) = \alpha\nu(abec)\beta = \alpha\mu(a)\mu(be \times c)\beta = 5 + 7 = 12.$$

- Similarly, for $w = a(bec)a(cd)$ we obtain :

$$I(abecacd) = \alpha\nu(abecacd)\beta = \alpha\mu(a)\mu(be{\times}c)\mu(a)\mu(d{\times}c)\beta = 23.$$

- Coalgebraic definition is for series and saves on complexity (no matrices)!

Extension to more local subsystems

For n=3 there are 4types of synchronizations: all the three and all couples of subsystems
Hence, 5 types of multi-events (including no synchronization)

$$\mathcal{A} = (A_1 \cap A_2 \cap A_3) \cup (A_1 \cap A_2) \times (A_3 \setminus (A_1 \cup A_2))^* \cup (A_1 \cap A_3) \times (A_2 \setminus (A_1 \cup A_3))^* \cup (A_2 \cap A_3) \times (A_1 \setminus (A_2 \cup A_3))^*$$

$$(A_1 \setminus (A_2 \cup A_3))^* \times (A_2 \setminus (A_1 \cup A_3))^* \times (A_3 \setminus (A_1 \cup A_2))^*$$

Equivalently,

$$\mathcal{A} = (A_1 \cap A_2 \cap A_3) \times (A_1 \cap A_2 \cap A_3) \times (A_1 \cap A_2 \cap A_3) \cup (A_1 \cap A_2) \times (A_1 \cap A_2) \times (A_3 \setminus (A_1 \cup A_2))^*$$

$$\cup (A_1 \cap A_3) \times (A_2 \setminus (A_1 \cup A_3))^* \times (A_1 \cap A_3) \cup (A_1 \setminus (A_2 \cup A_3))^* \times (A_2 \cap A_3) \times (A_2 \cap A_3)$$

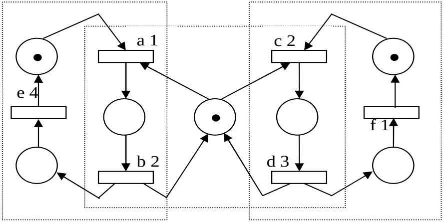
$$\cup (A_1 \setminus (A_2 \cup A_3))^* \times (A_2 \setminus (A_1 \cup A_3))^* \times (A_3 \setminus (A_1 \cup A_2))^*$$

$v = a \in A_1 \cap A_2 \cap A_3$ is here $a \times a \times a$
$v = a \times v_3$ is here $a \times a \times v_3$.

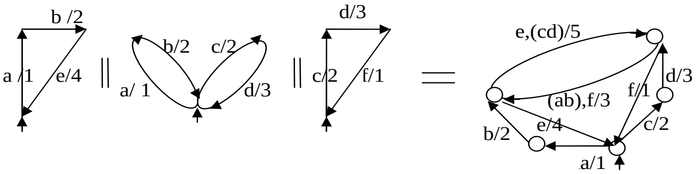
Interpretation in terms of timed Petri nets

- Synchronous products of (max,+) automata correspond to safe Timed Petri nets formed in a compositional way from safe timed state graphs (timed machines)
- Local automata correspond to marking graphs of timed state graphs (no synchronization: each transition has 1 upstream and 1 downstream place)
- Synchronous composition models synchronization of timed state graphs via synchronizing transitions

Example of timed Petri nets



Corresponding automaton model is below:



Outline

Timed automata and distributed interval automata

Timed automaton $\mathcal{A} = (S, A, C, t, s_0)$ with

- S . . . state set
- A . . . event set
- C . . . set of clocks
- $t \subseteq S \times A \times S \times EC \times 2^C$ . . . transition function

Transition labels: $Tr = \langle s, a, s', Cond, Z \rangle$, where
s origin, s' destination, a event label,
t can occur only if $Cond = TRUE$ and the clocks in Z are reset.
Syntax for enabling conditions (EC):
$c \equiv k$, where $c \in C$, $k \in R$, and $\equiv \in \{<, >, \leq, \geq, =\}$.
Extended states: $(s, c) \subseteq S \times R^{\|C\|}$,
with s state and
c the current values of clocks.

Distributed timed automata

Def. Composition of Timed automata
Synchronous product of timed automata
$R_i = (S_i, A_i, C_i, t_i, s_0^i), \; i = 1, \ldots, n$ is
$\|_{i=1^n} R_i = (S, A, C, t, s_0)$ with

- $S = \times_{i=1^n} S_i$
- $A = \bigcup_{i=1^n} A_i$
- $C = \bigcup_{i=1^n} C_i$
- $s_0 = (s_0^i)_{i=1}^n$
- $t \subseteq S \times A \times S \times EC \times 2^C$ such that $(s, a, s', \delta, \lambda) \in t$, iff $(s_i, a, s_i', \delta_i, \lambda_i) \in t_i$, where $s_i' = s_i$ for $a \in A_i$, $\delta = \delta_1 \wedge \delta_2$, and $\lambda = \lambda_1 \cup \lambda_2$.

Note. Regional construction is not compositional!

Interval Automata

Elementary classes of TA

- Product interval automata built by synchronous products of interval automata
- **Interval based alphabet:** $\Gamma = A \times IR$**,** with A finite alphabet and IR set of real intervals
- Definition of Interval automata
  Interval automata are automata $R = (S, \Gamma, t, I, F)$ over (symbolic) interval based alphabet.
- IA are timed automata with a single clock reset after every transitions

Product Interval Automata

- $R = (S, \Gamma, t, I, F)$ may also be viewed as weighted automaton with weights in a suitable interval semiring.
- Interval semiring: $(\mathbb{R}_{max} \times \mathbb{R}_{max}, \oplus, \otimes)$ with

$$(l_1, u_1) \otimes (l_2, u_2) = (l_1 + l_2, u_1 + u_2) \text{ and}$$

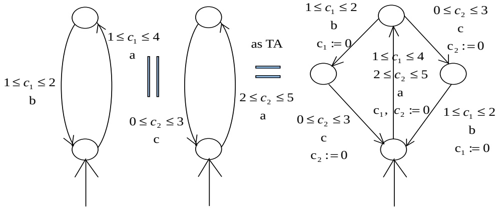
$$(l_1, u_1) \oplus (l_2, u_2) = (max(l_1, l_2), max(u_1, u_2))$$

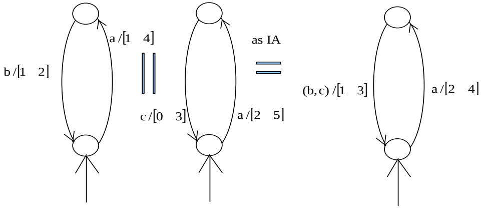
- Note that $\oplus$ is only used in composition, not in local IA (deterministic)!
- Dual addition also needed in the composition:

$$(l_1, u_1) \oplus' (l_2, u_2) = (max(l_1, l_2), min(u_1, u_2))$$

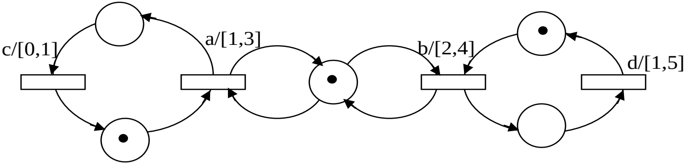
Composition of classes of timed automata

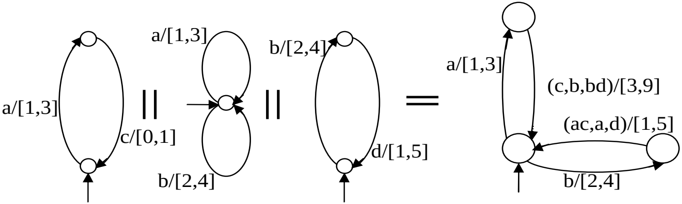
Definition. Synchronous products of interval automata.
Clocks are read and reset compatible with event distribution!





Example.





Example continued.

$$
\begin{aligned}
\mu(a) &= \mu_1(a) \otimes^t B_2(a) \otimes^t E_3 \oplus' B_1(a) \otimes^t \mu_2(a) \otimes^t E_3, \\
\mu(b) &= E_1 \otimes^t \mu_2(b) \otimes^t B_3(b) \oplus' E_1 \otimes^t B_2(b) \otimes^t \mu_3(b), \\
\mu((c,b,bd)) &= \mu_1(c) \otimes^t B_2(b) \otimes^t B_3(bd) \oplus B_1(c) \otimes^t \mu_2(b) \otimes^t B_3(bd) \oplus \\
&\quad B_1(c) \otimes^t B_2(b) \otimes^t \mu_3(bd), \\
\mu((ac,a,d)) &= \mu_1(ac) \otimes^t B_2(a) \otimes^t B_3(d) \oplus B_1(ac) \otimes^t \mu_2(a) \otimes^t B_3(d) \oplus \\
&\quad B_1(ac) \otimes^t B_2(a) \otimes^t \mu_3(d)
\end{aligned}
$$

Interpretation of extended words:
$w = acbdabdcbdac \in A^* \quad \rightarrow \quad w = a(c,b,bd)a(c,b,bd)b(ac,a,d)$
over $\mathcal{A}$.

Example coalgebraically.

Again, fundamental theorem gives for $\sigma \in \mathcal{A}^*$ and $v \in \mathcal{A}$

$$(l_1\|l_2\|l_3)(\sigma) = (l_1\|l_2\|l_3)(\sigma)(0) \oplus X(l_1\|l_2\|l_3)(\sigma)',$$

where

$$\{(l_1\|l_2\|l_3)(\sigma)\}' = (l_1\|l_2\|l_3)_{\sigma)(0)}(\sigma').$$

$(l_1\|l_2\|l_3)_v = (l_1)_{P_1(v)}\|(l_2)_{P_2(v)}\|(l_3)_{P_3(v)}$ and

$(l_1\|l_2\|l_3)[v] = l_1[P_1(v)] \otimes Bl_2[P_2(v)] \otimes Bl_3[P_3(v)] \oplus$

$Bl_1[P_1(v)] \otimes l_2[P_2(v)] \otimes Bl_3[P_3(v)] \oplus Bl_1[P_1(v)] \otimes B_2[P_2(v)] \otimes l_3[P_3(v)]$
with $\oplus$ replaced by $\oplus'$ for shared actions $v = a, b$

Outline

Concluding remarks

- Deterministic weighted automata as partial Mealy automata
- Composition of (max,+) automata
- Composition of Interval automata (PIA) and their properties
- Formulae for behavior of the synchronous product: algebraic vs. coalgebraic approach
- Supervisory control within behavioral framework
- Decentralized control of (classes) of timed automata