Coalgebraic Dynamic Quantum Logic

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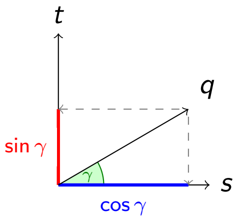
Outline

Quantum systems

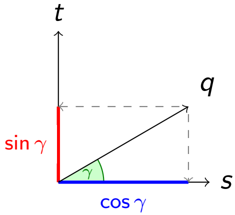
Quantum states have the following two properties:

- quantum states can be in a superposition
  $\Rightarrow$ probabilities
- tests (measurements) changes the quantum state
  $\Rightarrow$ modal operators



Quantum states

- A quantum state is a 1-dimensional ray in a Hilbert space.
- A test corresponds to a projection onto a closed subspace.
- Unitary operators are (reversible) rotations.
- We refer to both tests and unitaries as programs.



Quantum algorithms

- Shor's factoring algorithm (exponential speed-up).
- Grover's search algorithm (quadratic speed-up).
- Both algorithms are probabilistic!

Previous work

- (Baltag, Smets) A PDL-type quantum logic (with tests).
- (Leal Rodriguez) A coalgebraic PDL.
- (Abramsky) A coalgebraic framework, which can represent all physical symmetries.

Rough idea

- We take an arbitrary set of programs.
- We fix a set functor.
- We put restrictions on the coalgebra to obtain a quantum framework ("Hilbert space").

Coalgebraic quantum framework

- Set of states S.
- Set of tests $\mathcal{T} \subseteq \mathcal{P}S$.
- Set of unitaries $\mathcal{U}$.
- Set of programs $\Pi = \{P? \mid P \in \mathcal{T}\} \cup \mathcal{U}$.

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Given a coalgebra (S, σ), and a program q, then

- $\sigma(s)(q) = (p, t)$ means running program q on s leads to t with probability p and fails otherwise.
- $\sigma(s)(q) = 0$ means running q on s always fails.

Probabilistic modalities

We define a family of predicate liftings, for $q \in \Pi$ and $p \in [0,1]$, let

$$\lambda_S^{q,0}(Y) := \{\, \delta \in FS \mid \delta(q) \in (0,1] \times Y \,\}, \text{ and}$$
$$\lambda_S^{q,p}(Y) := \{\, \delta \in FS \mid \delta(q) \in [p,1] \times Y \,\}, (p > 0).$$

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We define the following labelled modalities:

$$[\![[q]\phi]\!] := \sigma^{-1} \circ \lambda_S^{q,0} [\![\phi]\!], \text{ and}$$
$$[\![[q]_{\geq p}\phi]\!] := \sigma^{-1} \circ \lambda_S^{q,p} [\![\phi]\!].$$

From coalgebra to functions

We define the following projections:

$$\pi_1 : \{0\} + (0,1] \times S \to [0,1], \text{ and}$$
$$\pi_2 : \{0\} + (0,1] \times S \rightharpoonup S.$$

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A coalgebra $\sigma : S \to FS$ associates with each $q \in \Pi$ a partial function

$$\overline{q} = \pi_2(\sigma(-)(q)) : S \rightharpoonup S.$$

Notation

- $\overline{\mathcal{T}?}(s) := \{\overline{P?}(s) \mid P \in \mathcal{T}\}$.
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- Quantum join; the closure of the span of P and Q:

$$P \sqcup Q := \sim(\sim P \cap \sim Q).$$

Axioms for testable properties

1. *Closure under arbitrary conjunctions*: $\bigcap \mathcal{T}' \in \mathcal{T}$ for any $\mathcal{T}' \subseteq \mathcal{T}$.
2. *Closure under orthocomplementation*: if $P \in \mathcal{T}$, then $\sim P \in \mathcal{T}$.
3. *Atomicity*: $\{s\} \in \mathcal{T}$ for any $s \in S$.

Axioms for tests

4. *Adequacy*: $\sigma(s)(P?) = (1, s)$ if $s \in P \in \mathcal{T}$.
5. *Repeatability*: $\overline{P?}(s) \in P$ whenever $\overline{P?}(s)$ is defined.
6. *Covering law*: if $\overline{P?}(s) \neq t \in P$, then $v \perp s$ for some $v \in \overline{\mathcal{T}?}(t) \cap P$.
7. *Self-adjointness*: for any $s, t \in S$

$$\pi_1(\sigma(\overline{P?}(s))(\{t\}?)) = \pi_1(\sigma(\overline{P?}(t))(\{s\}?)).$$

8. *Proper superposition*: $\overline{\mathcal{T}?}(s) \cap \overline{\mathcal{T}?}(t) \neq \emptyset$ for any $s, t \in S$.
9. If $P_0 \perp P_1$ ($P_0 \subseteq {\sim}P_1$), then for all $s \in S$

$$\pi_1(\sigma(s)(P_0 \sqcup P_1?)) = \pi_1(\sigma(s)(P_0?)) + \pi_1(\sigma(s)(P_1?)).$$

Axioms for unitary operators

10. *Reversibility and totality*: for every $s \in S$ there is a $t \in S$ such that $\sigma(s)(U) = (1, t)$ and for every $t \in S$ there is an $s \in S$ such that $\sigma(s)(U) = (1, t)$.
11. *Orthogonality preservation*: $s \perp t$ iff $\overline{U}(s) \perp \overline{U}(t)$ for any $s, t \in S$ and $U \in \mathcal{U}$.
12. *Mayet's condition*: there exist $U \in \mathcal{U}, P \in \mathcal{T}$ and $t, w \in S$ such that $\{\, \overline{U}(s) \mid s \in P \,\} \subsetneq P$, $t \perp w$, and, for every $s \in {\sim}{\sim}\{t, w\}$, $\overline{U}(s) = s$.

Conclusions and future work

We have shown that using coalgebras we can extend Baltag and Smets' quantum logic to a probabilistic setting.

- Axiomatize the logic.
- Explicitly add the tensor (for compound quantum systems).
- Investigate the nabla-operator ∇ (measurements).

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