On Coalgebraic Logic over Posets

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CMCS 2012, Tallinn, Estonia

Previous work

- Expressivity of Coalgebraic Logic over Poset
  (Kapulkin-Kurz-Velebil, CMCS2010)
- Finitary Functors: from Set to Preord and Poset
  (Balan-Kurz, CALCO2011)

Why posets?

Modal Logic	Want coalgebraic logic over posets to naturally generalize positive modal logic (Dunn 95).
Coalgebras	Looking at simulations instead of bisimulations? Then use posets as base.
Category Theory	Start with existing results on coalgebraic logics. Replace then Set by Poset.

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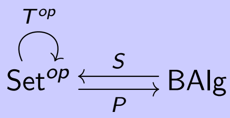
Work setting: enriched category theory over Poset

Coalgebraic logic for Set-functors

$$\mathsf{Set}^{op} \underset{P}{\overset{S}{\rightleftarrows}} \mathsf{BAlg}$$

- P maps a set to the BAlg of its subsets.
- S' associates to any BAlg the set of ultrafilters.

Coalgebraic logic for Set-functors



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Coalgebras:

- States: set X
- Dynamics: map $X \to TX$

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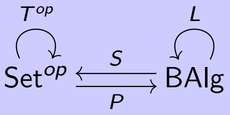

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Coalgebras:

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Abstract logic: (L, δ), where L : BAlg $\to$ BAlg is a functor and $\delta : LP \to PT^{op}$ a natural transformation.

Finitary coalgebraic logic: $L = PT^{op}S$ on finitely generated free BAlg, then canonically extended to all BAlg.

Coalgebraic logic for Poset-functors

$$\mathsf{Poset}^{op} \underset{P'}{\overset{S'}{\rightleftarrows}} \mathsf{DLat}$$

- Enriched adjunction
- P' maps a poset to the DLat of its upsets.
- S' associates to any DLat the poset of prime filters.

Coalgebraic logic for Poset-functors

$$T'^{op} \circlearrowright \mathsf{Poset}^{op} \underset{P'}{\overset{S'}{\leftrightarrows}} \mathsf{DLat}$$

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- T' locally monotone

Coalgebras:

- States: poset $\mathbb{X} = (X, \leq)$
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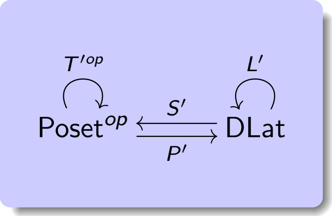
Coalgebraic logic for Poset-functors



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- L' locally monotone

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Finitary coalgebraic logic: $L' = P'T'^{op}S'$ on finitely generated free DLat, then canonically extended to all DLat.

Two logical connections...

$$T^{op} \circlearrowright \mathsf{Set}^{op} \underset{P}{\overset{S}{\rightleftarrows}}_{\perp} \mathsf{BAlg} \circlearrowleft L$$

$$T'^{op} \circlearrowright \mathsf{Poset}^{op} \underset{P'}{\overset{S'}{\rightleftarrows}}_{\perp} \mathsf{DLat} \circlearrowleft L'$$

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How to relate them?

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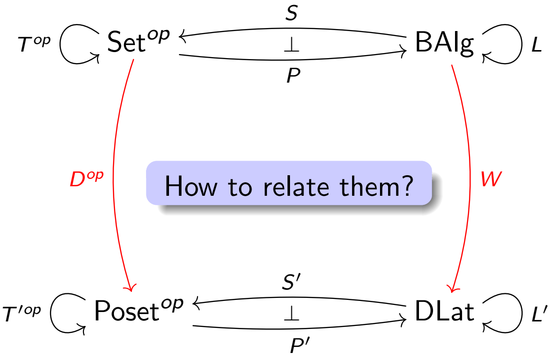
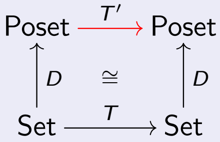


Coalgebraic side: extensions and posetifications

We fix a *Set*-functor T.

Definition (Balan-Kurz, CALCO2011)

An extension of T is a *locally monotone* functor $T' : \text{Poset} \to \text{Poset}$ such that $DT \cong T'D$.



An extension T' is called the posetification of T, if the above square exhibits T' as the Poset-enriched $\text{Lan}_D DT$.

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Fact: For each Set-functor T, the posetification $\mathrm{Lan}_D DT$ exists (this follows from general enriched category theory, because the discrete functor $D : \mathsf{Set} \to \mathsf{Poset}$ is dense).

Examples

1. $T = \mathrm{Id}$
   Then the discrete connected components functor DC and the upsets-functor $\mathcal{U}p$ are both extensions of T, while $\mathrm{Id} : \mathsf{Poset} \to \mathsf{Poset}$ is the posetification.

2. $T = \mathcal{P}$, the (finite) power-set functor
   Then its posetification is the (finitely generated) convex power-set functor, with the Egli-Milner order.

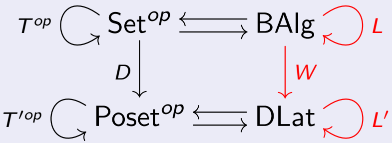
Relating abstract logics

T : Set $\to$ Set with logic (L, δ)

T' : Poset $\to$ Poset extension of T with logic (L', δ')

Definition

L' is a positive fragment of L if there is a natural transformation $L'W \Rightarrow WL$ commuting appropriately with δ and δ'.



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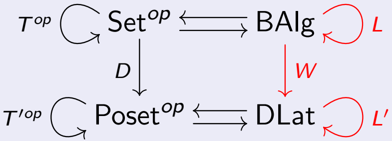


L' is the positive fragment of L if $L'W \Rightarrow WL$ is an isomorphism.

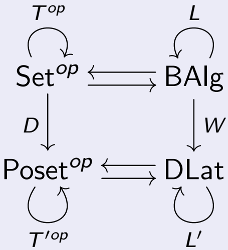
Main result

Theorem

Given the following:

- T *any* Set*-functor*
- T' *extension of* T
- (L, δ) *and* (L', δ') *the finitary logics of* T *and* T'
- T' *preserves coreflexive equalizers*

Then L' *is the positive fragment of* L*, i.e.* $WL \cong L'W$.



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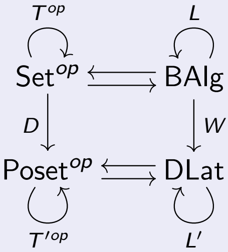

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Example

- $T = \mathcal{P}$ (finite) powerset functor

  Logics: LA is the BA generated by $\Box a$, for $a \in A$, wrt $\Box$ preserving finite meets.

  Semantics: $\delta_X : LPX \to P\mathcal{P}X, \;\; \Box a \mapsto \{b \in \mathcal{P}X \mid b \subseteq a\}$

- Posetification: $T' = \mathcal{P}_c$ (finitely generated) convex powerset functor

  Logics: $L'A$ is the DLat generated by $\Box a$ and $\Diamond a$, for all $a \in A$, wrt $\Box$ preserving finite meets, $\Diamond$ preserving finite joins, and

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Another example

- For $T = \mathrm{Id}$, the corresponding finitary logics is $L = \mathrm{Id}$ on BA, with trivial semantics $\delta : LP \to PT$.
- Extension: $T' = DC$ discrete connected components functor. T' does not preserve embeddings.

  Logics: L' is the constant functor to $\mathbb{2}$.
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What next?

More examples for future study

- Kripke functors $\mathcal{K} ::= \mathrm{Id} \mid K_X \mid \mathcal{K}_1 + \mathcal{K}_2 \mid \mathcal{K}_1 \times \mathcal{K}_2 \mid \mathcal{K}^A$
- $T = \mathcal{D}$ the (sub)distributions functor
- $T = \mathcal{Q}^2$ double contravariant functor

Current and future work

- Characterize those Poset-functors that arise as posetifications.
- Improve the present results using logical connections.
- Describe logics and their properties for extensions and posetifications.