Internal Modals for Coalgebraic Modal Logics

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Outline

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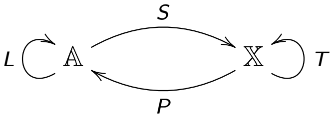
The big picture - Kurz, Abramsky...



$$\delta \colon LP \Rightarrow PT$$

1. $\mathbb{A}$ and $\mathbb{X}$ categories.
2. P and S are contravariant functors defining a dual adjunction.
3. L and T covariant endofunctors.
4. δ a natural transformation.

The base level - bivalent examples

$\mathbb{A}$: sets of formulae, probably with additional structure

e.g. **MSL**, **DL**, **BA**...

$\mathbb{X}$: sets of states or processes, possibly with additional structure

e.g. **Set**, **Top**, **Stone**, **Meas**...

P: maps a state space X to a collection of subsets of X (with the structure of an $\mathbb{A}$ object)

e.g. the powerset, the open sets, the clopen sets, the measurable sets...

S: maps an algebra of formulae A to a collection of logically consistent subsets of A (with the structure of an $\mathbb{X}$ object)

e.g. filters, prime filters, ultrafilters...

The dual adjunction

The dual adjunction between the contravariant functors P and S gives the base level semantics:

1. for every A in $\mathbb{A}$ and X in $\mathbb{X}$

   (**valuations**) $\{f : A \to P(X)\} \cong \{f^\flat : X \to S(A)\}$ (**theory maps**)

2. naturality means

$$x \in f(a) \Leftrightarrow a \in f^\flat(x)$$

In this context we call the dual adjunction a **logical connection**.

But where are the dynamics?

The systems we have so far are static.

We add dynamics using **coalgebras** for the functor $T\colon \mathbb{X} \to \mathbb{X}$

$$X \xrightarrow{\gamma} T(X)$$

The morphisms are the commuting squares

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\xi} \\ T(X) & \xrightarrow[T(f)]{} & T(Y) \end{array}$$

and this gives a category $\mathbf{CoAlg}(T)$.

Adding the modalities

Now we need to add modalities to our languages to represent the dynamics.

We do this using **algebras** for the functor $L \colon \mathbb{A} \to \mathbb{A}$

$$L(A) \xrightarrow{\alpha} A$$

The morphisms are the commuting squares

$$\begin{array}{ccc} L(A) & \xrightarrow{L(f)} & L(B) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow[f]{} & B \end{array}$$

and this gives a category $\mathbf{Alg}(L)$.

We have to be a little bit careful though!

Suppose we want to add $\Box$ to objects in **BA** to give us the **modal algebras**.

Naively we would take $L = id_{\mathbb{A}}$ and $\alpha = \Box \colon A \to A$, but this makes $\Box$ a **BA** homomorphism i.e.

$$\Box(a \wedge b) = \Box a \wedge \Box b \qquad ✓$$
$$\Box(a \vee b) = \Box a \vee \Box b \qquad ✗$$

We only want $\Box$ to preserve $\wedge$, so make it a morphism in **MSL**!

Do this using the adjunction $F \dashv U \colon$ **MSL** $\to$ **BA**, where U is the forgetful functor, and F is the free construction.

Semantics for the modalities

The semantics of the modalities are given by a natural transformation $\delta\colon LP \Rightarrow PT$.

Using this we can define a functor $\tilde{P}\colon \mathbf{CoAlg}(T) \to \mathbf{Alg}(L)$ given by

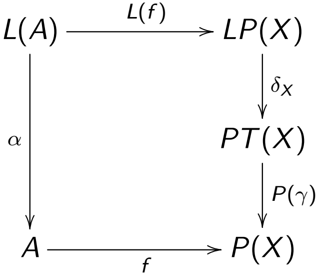
$$X \xrightarrow{\gamma} T(X) \quad \mapsto \quad LP(X) \xrightarrow{\delta_X} PT(X) \xrightarrow{P(\gamma)} P(X)$$

Valuations are then L-algebra morphisms

$$(A, \alpha) \xrightarrow{f} \tilde{P}(X, \gamma)$$

Expanding...

A valuation is thus a square



So it is a valuation in the base language constrained to interact correctly with the modalities.

Observation

The collection of all valuations for an L-algebra (A, α) forms the comma category

$$((A, \alpha) \downarrow \tilde{P})$$

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A result from Pavlovic, Mislove, and Worrell

Using the logical connection a valuation

$$(A, \alpha) \xrightarrow{f} \tilde{P}(X, \gamma)$$

can be redrawn as

$$\begin{array}{ccc} X & \xrightarrow{f^\flat} & S(A) \\ & & \big\downarrow{\scriptstyle S(\alpha)} \\ {\scriptstyle \gamma}\big\downarrow & & SL(A) \\ & & \big\uparrow{\scriptstyle \delta^*_A} \\ T(X) & \xrightarrow[T(f^\flat)]{} & TS(A) \end{array}$$

where $\delta^* \colon TS \Rightarrow SL$ is the dual or transpose of δ.

Models for an L-algebra

Define the category $\mathbf{Mod}(A, \alpha)$ of **models** for the L-algebra (A, α) by:

1. The objects are pairs $((X, \gamma), f : X \to S(A))$ such that the previous diagram commutes. Call f a **theory map**.
2. The morphisms are T-coalgebra morphisms such that if $g : ((X_1, \gamma_1), f_1) \to ((X_2, \gamma_2), f_2)$ then $f_1 = f_2 \circ g$.

The logical connection means that $\mathbf{Mod}(A, \alpha)$ is dually isomorphic to the comma category $((A, \alpha) \downarrow \tilde{P})$.

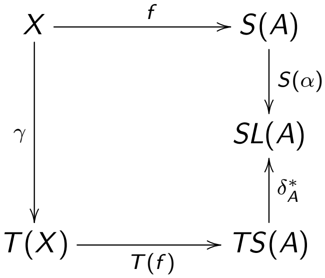
Observations

1. If (A, α) is the initial L-algebra then for every T-coalgebra there is a unique theory map making it a model.
2. For a general (A, α) there may be some T-coalgebras for which no theory maps exist that make them into models.

An idea from Kripke semantics

Kripke semantics has the concept of a **canonical model** - a model constructed from the syntax of the language itself.

We can generalise this by considering models with f **injective**



Internal models for an L-algebra

Given a class M of monomorphisms in $\mathbb{X}$, we define the category $\mathbf{IntMod}_M(A, \alpha)$ to be the full subcategory of $\mathbf{Mod}(A, \alpha)$ where the theory maps are in M, and write

$$G \colon \mathbf{IntMod}_M(A, \alpha) \to \mathbf{Mod}(A, \alpha)$$

for the corresponding inclusion functor.

The objects of $\mathbf{IntMod}_M(A, \alpha)$ we call **internal models** of (A, α).

The parameterisation by M allows a restriction to say embeddings in **Meas**, as these are preserved by the Giry functor, which will be useful later.

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An adjoint functor theorem

When is there a functor $\tilde{S}\colon \mathbf{Alg}(L) \to \mathbf{CoAlg}(T)$ such that it forms a dual adjunction with $\tilde{P}$?

Recall the proof of The Freyd Adjoint Functor Theorem - we need to construct initial objects in the comma categories $((A, \alpha) \downarrow \tilde{P})$ - but this is the same as final objects in $\mathbf{Mod}(A, \alpha)$!

Sufficient conditions are that for all L-algebras (A, α) the following hold:

1. for all X in $\mathbf{Mod}(A, \alpha)$ there exists a $g\colon X \to G(I)$ for some object I in $\mathbf{IntMod}_M(A, \alpha)$,
2. $\mathbf{IntMod}_M(A, \alpha)$ has a final object.

Expressivity

For two models X_1, X_2 in $\mathbf{Mod}(A,\alpha)$, and $x_1 \in X_1$, $x_2 \in X_2$, we say x_1 and x_2 are **behaviourally equivalent** if there exists in $\mathbf{Mod}(A,\alpha)$ a cospan

$$X_1 \xrightarrow{f_1} X_3 \xleftarrow{f_2} X_2$$

such that $f_1(x_1) = f_2(x_2)$.

We say (A,α) is **expressive**, if for all models in $\mathbf{Mod}(A,\alpha)$, states have the same theories if and only if they are behaviourally equivalent.

Sufficient conditions for expressivity

If we choose M to be some subclass of the class of monos in $\mathbb{X}$ with injective underlying functions, then sufficient conditions for expressiveness are:

1. for all X in $\mathbf{Mod}(A, \alpha)$ there exists a $g\colon X \to G(I)$ for some object I in $\mathbf{IntMod}_M(A, \alpha)$,
2. for every pair I_1, I_2 in $\mathbf{IntMod}_M(A, \alpha)$ there is a cospan

$$I_1 \xrightarrow{f_1} I_3 \xleftarrow{f_2} I_2$$

in $\mathbf{IntMod}_M(A, \alpha)$.

These are weaker requirements than for the adjoint functor theorem.

A characterisation of expressivity

Given:

1. some mild assumptions about the structure of $\mathbf{Mod}(A, \alpha)$,
2. $\mathbb{X}$ has binary coproducts,
3. the class M of monomorphisms is precisely the class of morphisms with injective underlying functions,

then (A, α) is expressive for $\mathbf{Mod}(A, \alpha)$ if and only if:

1. for all X in $\mathbf{Mod}(A, \alpha)$ there exists a $g : X \to G(I)$ for some object I in $\mathbf{IntMod}_M(A, \alpha)$,
2. for every pair I_1, I_2 in $\mathbf{IntMod}_M(A, \alpha)$ there is a cospan

$$I_1 \xrightarrow{f_1} I_3 \xleftarrow{f_2} I_2$$

in $\mathbf{IntMod}_M(A, \alpha)$.

Factorisation systems in the base category

Suppose that a class E of morphisms in $\mathbb{X}$ exists such that $\mathbb{X}$ has a factorisation system (E, M).

Also suppose (Klin, Jacobs and Sokolova)

$$m \in M \Rightarrow \delta_A^* \circ T(m) \in M$$

then the following hold:

1. condition 1 of the previous theorems,
2. the forgetful functor $U \colon \mathbf{IntMod}_M(A, \alpha) \to \mathbb{X}$ detects small colimits.

Corollaries

If $\mathbb{X}$ has a factorisation system (E, M), and

$$m \in M \Rightarrow \delta_A^* \circ T(m) \in M$$

then

1. $\mathbb{X}$ has binary coproducts $\Rightarrow (A, \alpha)$ is expressive,
2. $\mathbb{X}$ is M-wellpowered and has small coproducts $\Rightarrow$ $\mathbf{IntMod}_M(A, \alpha)$ has a final object,

If this last result holds for all (A, α), then there is a dual adjunction between $\mathbf{Alg}(L)$ and $\mathbf{CoAlg}(T)$.

Example 1

Take:

1. T to be $\mathcal{P}_f : \mathbf{Set} \to \mathbf{Set}$ the finite powerset functor,
2. L to add an operator $\Box$ to the objects of **BA**,
3. the factorisation system (Surjective, Injective) in **Set**,
4. the obvious choice for δ (given by the predicate lifting in **MSL**),

then there is a dual adjunction between modal algebras and image-finite transition systems.

This then yields corresponding expressivity results for all modal algebras - not just the initial.

Example 2

Take:

1. T to be the Giry functor on **Meas**,
2. L to add a countable set of modalities L_r for $r \in \mathbb{Q} \cap [0,1]$ to the objects of **MSL** (where $L_r\phi$ means ϕ is true with probability at least r),
3. the factorisation system (Surjective, Embeddings) in **Meas**,
4. the obvious choice for δ (given by the predicate liftings in **Pos**),

then there is a dual adjunction between probabilistic modal algebras and Markov processes.

Again this yields corresponding expressivity results for all probabilistic modal algebras - not just the initial.

Outline

Future work

As noted earlier, all the examples in this talk are bivalent.

This is because the logical connection arises from a two element **dualising object**.

In recent work Kurz and Velebil have looked at logical connections in an enriched setting with a more general notion of dualising object.

I am currently investigating whether my expressivity results can be extended to such enriched logical connections.

Related work

The idea that internal models might be a fruitful thing to study follows from the work in:

Jacobs, B., Sokolova, A.: Exemplaric Expressivity of Modal Logics. Journal of Logic and Computation 20(5) (2010) 1041–1068

Klin, B.: Coalgebraic modal logic beyond sets. Electronic Notes in Theoretical Computer Science 173 (2007) 177–201

Questions

Any questions?