

# Coinitial semantics for redecoration of triangular matrices

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Simple inductive types—*W*-types—are characterized categorically as initial algebras of a polynomial functor. Dually, *co*inductive types are characterized as terminal *co*algebras of polynomial functors. In the case of *co*inductive types, the meta-theoretic notion of equality is not adequate: thus, the notion of *bisimulation* was introduced by Aczel [1].

The characterization of inductive types as initial algebras has been extended to *heterogeneous*—also called *nested*—inductive data types, e.g., the type of  $\lambda$ -terms, in various formulations [3, 4]. The main goal of those works is to characterize not only the data type via a universal property, but rather the data type *equipped with a well-behaved substitution operation*.

In the present work we study a specific *co*inductive *heterogeneous* data type—the type family *Tri* of infinite triangular matrices—and its *redecoration* operation: the codata type is parametrized by a fixed set  $E$  for entries not on the diagonal, and indexed by another, *variable*, set  $A$  for entries on the diagonal. The respective types of its specifying destructors *top* and *rest* are given in Figure 1, together with the destructors for the *co*inductively defined bisimilarity relation on it. Equipped with the redecoration operation, the type *Tri* is shown by Matthes and Picard [5] to constitute what they call a “weak constructive comonad”.

In this work, we first identify those weak constructive comonads as an instance of the more general notion of *relative comonad*. Indeed, a weak constructive comonad is precisely a comonad relative to the functor  $\text{eq} : \text{Set} \rightarrow \text{Setoid}$  from the category of sets to that of setoids that is left adjoint to the forgetful functor.

Afterwards, we characterize the codata type *Tri*, equipped with the *co*substitution operation of redecoration, as a terminal object of some category. For this, we dualize the approach by Hirschowitz and Maggesi [4], who characterize the *heterogeneous* inductive type of lambda terms—equipped with a suitable substitution operation—as an initial object in a category of algebras for the

$$\begin{array}{cc}
 \frac{t : \text{Tri}(A)}{\text{top}_A(t) : A} & \frac{t : \text{Tri}(A)}{\text{rest}_A(t) : \text{Tri}(E \times A)} \\
 \\
 \frac{t \sim t'}{\text{top}(t) = \text{top}(t')} & \frac{t \sim t'}{\text{rest}(t) \sim \text{rest}(t')}
 \end{array}$$

**Fig. 1.** Destructors and bisimilarity for the *co*inductive family of setoids *Tri*

signature of lambda terms. In their work, the crucial notions are the notion of monad and, more importantly, *module over a monad*. It turns out that more work than a simple dualization is necessary for two reasons:

- the lambda calculus can be seen as a monad on sets and thus, in particular, as an endofunctor. The codata type `Tri`, however, associates to any *set* of potential diagonal elements a *setoid* of triangular matrices. We thus need a notion of comonad whose underlying functor is not necessarily endo: the already mentioned *relative* comonads;
- the category-theoretic analysis of the destructor `rest` is more complicated than that of the heterogeneous constructor of abstraction of the lambda calculus.

Finding a suitable categorical notion to capture the destructor `rest` and, more importantly, its interplay with the comonadic redecoration operation on `Tri`, constitutes the main contribution of the present work. These rather technical details shall not be explained in this extended abstract.

Once we have found such a categorical notion, we can use it to give a definition of a “coalgebra” for the signature of infinite triangular matrices, together with a suitable notion of *morphism* of such coalgebras. We thus obtain a category of coalgebras for that signature. Any object of this category comes with a comonad relative to the aforementioned functor  $\text{eq} : \text{Set} \rightarrow \text{Setoid}$  and a suitable comodule over this comonad, modeling in some sense the destructor `rest`. Our main result then states that this category has a terminal object built from the codata type `Tri` and its destructor `rest`, which are seen as a relative comonad and a comodule over that relative comonad, respectively. This universal property of cointiality characterizes not only the codata type of infinite triangular matrices but also the **bisimilarity** relation on it as well as the **redecoration** operation.

All our definitions, examples, and lemmas have been implemented in the proof assistant `Coq`. The `Coq` source files and HTML documentation are available on the project web page [2]. While parts of our work seem to be specific to the particular codata type `Tri`, we believe that our work proves the suitability of the notion of relative (co)monads and (co)modules thereover for a categorical characterization of coinductive data types.

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