# Coalgebras with internal moves

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# Contents

# Labelled transition system with $\tau$-label

Put $\Sigma_\tau = \Sigma + \{\tau\}$. LTS with $\tau$-label:

$$X \to \mathcal{P}(\Sigma_\tau \times X) \cong \mathcal{P}(\Sigma \times X + X)$$

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**Definition (one of the variants)**

*Weak bisimulation* on $\alpha : X \to \mathcal{P}(\Sigma_\tau \times X) \stackrel{df}{=}$ strong bisimulation on $\alpha^*$

Here,

$$x \xrightarrow{a}_{\alpha^*} x' \iff x \xrightarrow{\tau^*}_{\alpha} \circ \xrightarrow{a}_{\alpha} \circ \xrightarrow{\tau^*}_{\alpha} x' \text{ for visible } a \in \Sigma,$$

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# Non-deterministic automata with $\varepsilon$-moves

$\varepsilon$-NA's coalgebraically:

$$\alpha : X \to \mathcal{P}(\Sigma_\varepsilon \times X + 1) \cong \mathcal{P}(\Sigma \times X + 1 + X)$$

## Definition

*Weak trace semantics:* $\mathrm{tr}_\alpha : X \to \mathcal{P}(\Sigma^*)$:

$w \in \mathrm{tr}_\alpha(x) \iff w = \varepsilon$ and $x$ is final or
$w = a_1 \ldots a_n$ and $x(\stackrel{\varepsilon}{\to})^* \circ \stackrel{a_1}{\to} \circ (\stackrel{\varepsilon}{\to})^* \ldots (\stackrel{\varepsilon}{\to})^* \circ \stackrel{a_n}{\to} \circ (\stackrel{\varepsilon}{\to})^* x'$
with $x'$ final.

# And many more...

- Segala systems,
- fully probabilistic systems,
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Coalgebras with internal moves

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$$\frac{\text{Coalgebras with internal moves}}{X \to T(FX + X)}$$

# Where the problem starts

Weak trace semantics is simple to handle:

1. consider a system with internal moves $X \to T(FX + X)$,
2. treat it as if it had only visible labels $X \to TF'X$ for $F'X = FX + X$,
3. find the standard trace semantics morphisms for all-visible-steps $TF'$-coalgebra and remove all occurrences of $\varepsilon$-label from it.

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## Problems

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**Problems**

- How to deal with weak bisimulation?
- What are structural properties of systems with internal moves?

# Hiding internal moves inside a monadic structure

Put $F_\tau = F + \mathcal{I}d$ and consider the functor

$$T(F + \mathcal{I}d) = TF_\tau$$

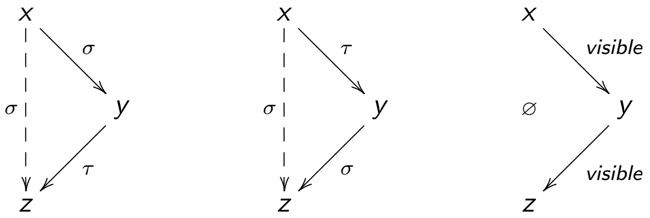
**Strategy 1**

If $\mathcal{K}l(T)$ admits zero morphisms (i.e. $0 \cdot f = g \cdot 0 = 0$) and $F : \mathsf{C} \to \mathsf{C}$ lifts to $\mathcal{K}l(T)$ then we may impose a monadic structure on $TF_\tau$.

# Strategy 1: example

Put $T = \mathcal{P}$ and $F_\tau = \Sigma_\tau \times \mathcal{I}d$. The LTS functor $\mathcal{P}(\Sigma_\tau \times \mathcal{I}d)$ carries a monadic structure whose composition in the Kleisli category is given as follows. For $f : X \to \mathcal{P}(\Sigma_\tau \times Y)$ and $g : Y \to \mathcal{P}(\Sigma_\tau \times Z)$ we have $g \cdot f : X \to \mathcal{P}(\Sigma_\tau \times Z)$:

$$g{\cdot}f(x) = \{(\sigma, z)) \mid x \stackrel{\sigma}{\to}_f y \stackrel{\tau}{\to}_g z \text{ or } x \stackrel{\tau}{\to}_f y \stackrel{\sigma}{\to}_g z \text{ for some } y \in Y\}.$$



# Hiding internal moves inside a monadic structure

**Strategy 2**

If $F$ admits all free algebras and lifts to $\mathcal{K}l(T)$ then we may embed the functor $TF_\tau$ into the monad $TF^*$.

# Strategy 2: example 1

Let $T = \mathcal{P}$ and $F = \Sigma \times \mathcal{I}d$:

$$\underline{X \to \mathcal{P}(\Sigma_\tau \times X)}$$

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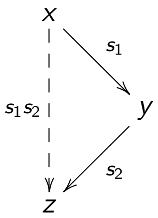
The composition $\cdot$ in $\mathcal{K}l(\mathcal{P}(\Sigma^* \times \mathcal{I}d))$ is given by the following formula. For $f : X \to \mathcal{P}(\Sigma^* \times Y)$ and $g : Y \to \mathcal{P}(\Sigma^* \times Z)$ we have $g \cdot f : X \to \mathcal{P}(\Sigma^* \times Z)$:

$g \cdot f(x) = \{(s_1 s_2, z) \mid x \stackrel{s_1}{\to}_f y \stackrel{s_2}{\to}_g z$ for some $y \in Y$ and $s_1, s_2 \in \Sigma^*\}$.



# Strategy 2: example 2

Let $T = \mathcal{P}$ and $F = \Sigma \times \mathcal{I}d + 1$:

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For $f : X \to \mathcal{P}(\Sigma^* \times Y + \Sigma^*)$ and $g : Y \to \mathcal{P}(\Sigma^* \times Z + \Sigma^*)$ we have $g \cdot f : X \to \mathcal{P}(\Sigma^* \times Z + \Sigma^*)$:

$$\begin{aligned} g \cdot f(x) = &\{(s_1 s_2, z) \mid x \xrightarrow{s_1}_f y \xrightarrow{s_2}_g z \text{ for some } y \in Y \text{ and } s_1, s_2 \in \Sigma^*\} \cup \\ &\{s_1 s_2 \mid x \xrightarrow{s_1}_f y \text{ and } y \downarrow_g s_2 \text{ for some } y \in Y\} \cup \\ &\{s_1 \mid x \downarrow_f s_1\}. \end{aligned}$$

# Section summary

Given $TF_\tau$ where $T$ is a monad and $F$ a functor which lifts to $\mathcal{K}l(T)$ we can:

Impose a monadic structure on $TF_\tau$ (provided that $\mathcal{K}l(T)$ admits zero morphisms)

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$$\frac{\text{Systems with internal moves}}{\text{Coalgebras over a monad}}$$

# Contents

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**Remark**

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Weak bisimulation on $\alpha$ is defined as strong bisimulation on a saturated structure $\alpha^*$.

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For $T = \mathcal{P}(\Sigma^* \times \mathcal{I}d)$

$$\alpha^{\circledast} = 1 \vee \alpha \vee \alpha^2 \vee \dots$$

**Remark**

$\alpha^\bigstar / \alpha^{\circledast}$ is the reflexive and transitive closure of $\alpha$ w.r.t. the order on hom-sets and Klesli composition.

# Order saturation monad - definition

A monad $T$ whose Kleisli category is order-enriched is called *ordered $*$-monad* or *ordered saturation monad* provided that in $\mathcal{K}l(T)$ for any morphism $\alpha : X \multimap X$ there is a morphism $\alpha^* : X \multimap X$ satisfying the following conditions:

(a) $1 \leqslant \alpha^*$,

(b) $\alpha \leqslant \alpha^*$,

(c) $\alpha^* \cdot \alpha^* \leqslant \alpha^*$,

(d) if $\beta : X \multimap X$ satisfies $1 \leqslant \beta$, $\alpha \leqslant \beta$ and $\beta \cdot \beta \leqslant \beta$ then $\alpha^* \leqslant \beta$,

(e) extra technical condition.

# Weak bisimulation

Let $T$ be an order saturation monad. Intuitively, a saturation $(-)^*$ can be thought of as least fixed point of:

$$\alpha^* = \mu x.(1 \vee x \cdot \alpha)$$

**Definition**

A *weak bisimulation* on $\alpha$ is defined as a strong bisimulation on $\alpha^*$.

This treatment encompasses:

- various types of LTS with algebraic structure on labels,
- Segala systems (viewed as $\mathcal{CM}(\Sigma_\tau \times \mathcal{I}d)$-coalgebras),
- ...

# What is done in the paper regarding WB?

Recall that given an LTS $\alpha : X \to \mathcal{P}(\Sigma_\tau \times X)$ we may saturate the structure in two different ways:

For $T = \mathcal{P}(\Sigma_\tau \times \mathcal{I}d)$

$$\alpha^\bigstar = 1 \vee \alpha \vee \alpha^2 \vee \ldots$$

For $T = \mathcal{P}(\Sigma^* \times \mathcal{I}d)$

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However, weak bisimulations for both approaches coincide.
Consider $\alpha : X \to TF_\tau X$

$TF_\tau$ $\qquad$ $TF^*$

$\alpha^\bigstar$ $\qquad$ $\alpha^\circledast$

We compare strong bisimulations for $\alpha^\bigstar$ and $\alpha^\circledast$.

# Contents

# $\varepsilon$-NA trace semantics: top-down approach

The classical procedure:

1. consider $X \to \mathcal{P}(\Sigma_\varepsilon \times X + 1)$,
2. treat all labels as visible labels,
3. find the standard trace semantics morphisms for all-visible-steps $\mathcal{P}(\Sigma_\varepsilon \times X + 1)$-coalgebra [by final semantics in $\mathcal{K}l(\mathcal{P})$] and remove all occurrences of $\varepsilon$-label from it.

# $\varepsilon$-NA trace semantics: bottom-up approach

1. $$\frac{X \to \mathcal{P}(\Sigma_\varepsilon \times X + 1)}{X \to \mathcal{P}(\Sigma^* \times X + \Sigma^*)}$$
2. the Klesli category for $\mathcal{P}(\Sigma^* \times X + \Sigma^*)$ is **Cppo**-enriched,
3. for any $\alpha : X \multimap X$ in $\mathcal{K}l(\mathcal{P}(\Sigma^* \times X + \Sigma^*))$ put $\mathsf{tr}_\alpha : X \multimap \varnothing$ ($\mathsf{tr}_\alpha : X \to \mathcal{P}(\Sigma^*)$):

$$\mathsf{tr}_\alpha = \mu x.x \cdot \alpha.$$

# Weak trace semantics for coalgebras

Let $T$ be a monad on C. A coalgebraic weak trace semantics is defined as an operator

$$\mathsf{tr}_{(-)} : Hom_{\mathcal{K}l(T)}(X, X) \to Hom_{\mathcal{K}l(T)}(X, 0)$$

such that:

- $\mathsf{tr}_\alpha = \mathsf{tr}_\alpha \cdot \alpha$,
- $\mathsf{tr}_\alpha$ is uniform.

## Why uniformity?

- strong bisimilarity implies weak trace semantics,
- uniqueness of a uniform fixed point operator is directly connected to the initial algebra = final coalgebra coincidence in the base category C.

# Contents

# Can we compare strong and weak bisimulation together with trace semantics?

Assume in $\mathcal{K}l(T)$ all suprema exist and the non-empty ones are preserved by the composition (e.g. in $\mathcal{K}l(\mathcal{P}(\Sigma^* \times \mathcal{I}d + \Sigma^*))$ we have so). In this case

$$\alpha^* = \mu x.(1 \vee x \cdot \alpha) \qquad \mathsf{tr}_\alpha = \mu x. x \cdot \alpha.$$

**Theorem**

*If $T$ is a monad as above then:*

$$\textit{strong bisimilarity} \implies \textit{weak bisimilarity} \implies \textit{weak trace equivalence.}$$

Thank you!

# Bilbiography

1. T. Brengos, *Weak bisimulations for coalgebras over ordered enriched monads*, 2013 (http://arxiv.org/abs/1310.3656)
2. Simpson, A., Plotkin, G.: Complete Axioms for Categorical Fixed-point Operators. LICS, 2000.