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Coalgebraic Games

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Games

- A wide gamut of interactions and other dynamic phenomena are described using game-based metaphors
- Various notions of games arise in the literature, *e.g.* in semantics, model checking, economics ...
- Games of perfect/non-perfect information, sequential/concurrent games, ...
- Many concepts, no standard definitions: move, position, play, turn, winning condition, payoff function, strategy, ...
- Sometimes complex definitions ...
- A unifying framework is called for ...

Coalgebraic games

The coalgebraic framework provides a convenient conceptual setting for describing games:

games	are defined as	elements of a final coalgebra
game operations	are defined as via (generalized)	final morphisms corecursion schemata

[Honsell-L. CALCO 2009, Lescanne-Perrinel 2012, Abramsky-Winschel 2013]

Coalgebraic games are 2-player games of perfect information

- 2-players games, Left (L) and Right (R)
- games have positions
- a game is identified with its initial position
- at any position, there are moves for L and R, taking to new positions of the game
- plays are possibly infinite
- perfect knowledge: all positions are public to both players
- payoff, ...

Benefits of the coalgebraic framework

- smooth definitions of games as elements of a final coalgebra and of game operations as final morphisms;
- games are represented as minimal graphs up-to bisimilarity, abstracting away from superficial features of positions;
- coalgebraic games subsume games of various contexts: Conway's (loopy) games, Game Semantics, Economics, ...
- the coalgebraic approach allows for a fine analysis of game operations, shedding light on the relationships between categorical constructions of Game Semantics and categories of games in the style of Joyal;
- coalgebras offer an alternative framework for Game Semantics, where basic notions admit simpler definitions, and they allow to factorize the steps bringing from sequential to concurrent games.

Plan of the talk

1. The framework of coalgebraic games:
    - coalgebraic games and strategies
    - coalgebraic operations
    - coalgebraic equivalences
2. Categories of coalgebraic games:
    - generalizing Joyal's category of Conway's games to infinite loopy games
    - Game Semantics, coalgebraically
3. From sequential to concurrent games: coalgebraic multigames

1. The framework of coalgebraic games

Coalgebraic games

Definition (Coalgebraic Games [Honsell-L. 2009])

- Let $\mathcal{A}$ be a set of atomic moves with functions:
  (i) $\mu : \mathcal{A} \rightarrow \mathcal{N}$ yielding the name of the move (for a set $\mathcal{N}$ of names),
  (ii) $\lambda : \mathcal{A} \rightarrow \{L, R\}$ yielding the player who has moved.
- Let $F_{\mathcal{A}} : \mathrm{Set}^* \rightarrow \mathrm{Set}^*$ be the functor defined by

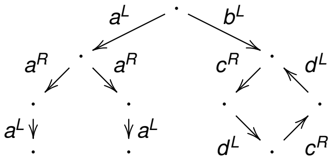
$$F_{\mathcal{A}}(A) = \mathcal{P}_{<\kappa}(\mathcal{A} \times A) \ .$$

- Let $(\mathbf{G}_{\mathcal{A}}, id)$ be the final $F_{\mathcal{A}}$-coalgebra.

A coalgebraic game is an element x of the carrier $\mathbf{G}_{\mathcal{A}}$ of the final coalgebra.

Coalgebraic games vs games as graphs

- Often games are represented as graphs, where nodes are positions and arcs are moves.
- Coalgebraic games are the minimal graphs up-to bisimilarity.
- Our coalgebraic approach is motivated by the fact that the existence of winning/non-losing strategies (and other equivalences on games) is preserved under graph bisimilarity of games.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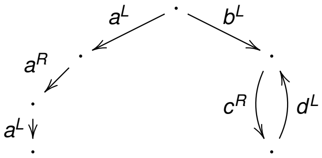
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Plays

Definition (Plays)

- A play on a game x_0 is a (finite or infinite) stream of pairs on $\mathcal{A} \times \mathbf{G}_{\mathcal{A}}$, $s = \langle a_1, x_1 \rangle \ldots$ such that $\forall n \geq 0.\ \langle a_{n+1}, x_{n+1} \rangle \in x_n$.
- A finite play is winning for the player who performs the last move.
- Infinite plays s can be taken to be winning for L/R ($\nu(s) = 1/-1$) or draws ($\nu(s) = 0$), ν payoff function.

Strategies

Strategies for a given player are partial functions on finite alternating plays ending with a position where the player is next to move, yielding (if any) a pair consisting of "a move of the given player together with a next position":

Definition

- A partial strategy σ for LI (L acting as player I) is a partial function $\sigma : FPlay_x^{LI} \to \mathcal{A} \times \mathbf{G}_{\mathcal{A}}$ s.t. $\sigma(s) = \langle a, x' \rangle \implies \lambda a = L \ \wedge \ s\langle a, x' \rangle \in FPlay_x$. Similarly, one can define strategies for players LII, RI, RII.
- A total strategy is a partial strategy s.t. $\exists \langle a, x' \rangle . \ (s\langle a, x' \rangle \in FPlay_x \ \wedge \ \lambda a = L) \implies s \in dom(\sigma)$.
- A strategy is winning for LI if, when played against any counterstrategy, it always produces winning plays for L.
- A strategy is non-losing for LI if, when played against any counterstrategy, it always produces non-losing plays for L.

Coalgebraic characterization of strategies

(Total) strategies for LI, LII:

- Let S : Set* $\times$ Set* $\rightarrow$ Set* $\times$ Set* be the functor:

$$S_L(X, Y) = \langle S_{LI}Y, S_{LII}X\rangle \ ,$$

  where $S_{LI}Y = \mathcal{A}_L \times Y, \quad S_{LII}X = \mathcal{P}_{<\kappa}(\mathcal{A}_R \times X)$,
  and $\mathcal{A}_L$ ($\mathcal{A}_R$) is the set of L-moves (R-moves).

- The total strategies for LI and LII on coalgebraic games in $\mathbf{G}_{\mathcal{A}}$ are the elements of the final S-coalgebra $\mathbf{S} = \langle \mathbf{S}_{LI}, \mathbf{S}_{LII}\rangle$.

- $\sigma \in \mathbf{S}_{LI}$ ($\mathbf{S}_{LII}$) is a strategy on the game x iff there exists a simulation between σ and x, $\sigma \leq x$.

Strategies for RI, RII are defined similarly.

Interesting classes of coalgebraic games

- Mixed games: the whole class of coalgebraic games, where plays can be winning or draws.
- Fixed games: the subclass of games where all plays are winning for one of the players.
- Polarized games: the subclass of games where
  - each position is marked as L or R, that is only L or R can move from that position,
  - R starts,
  - L/R positions strictly alternate.

Fixed and polarized games typically arise in Game Semantics.

Conway games

- Conway (wellfounded) games are inductively defined as pairs of sets $x = (X^L, X^R)$, where X^L (X^R) is the set of next positions to which L (R) can move.
- Such games are purely positional, no move names.
- In [Berlekamp-Conway-Guy82] (non-wellfounded) loopy games are defined as graphs of positions.
- In our coalgebraic setting, we represent possibly non-wellfounded Conway games as elements of the final coalgebra $\mathbf{G}_{\mathcal{A}}$, for $\mathcal{A} = \{a^L, a^R\}$, where $\mu a^L = \mu a^R = a$, $\lambda a^L = L$ and $\lambda a^R = R$.
- This extends the original set-theoretical definition of Conway.
- Coalgebraic games correspond to loopy games up-to graph bisimilarity.

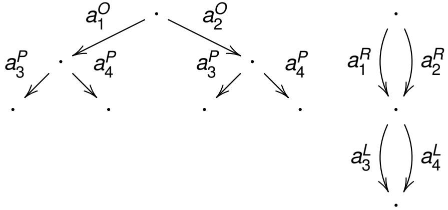
Games à la [Abramsky-Jagadeesan-Malacaria]

AJM-game $G = (M_G, \lambda_G, P_G, W_G)$

- M_G set of moves
- $\lambda_G : M_G \to \{O, P\}$
- O and P move in alternation
- O starts the game
- P_G set of legal positions, finite alternating sequences of moves
- W_G winning condition: a finite play is winning for a player if there are no moves for the other player, infinite plays are fixed to be winning either for O or for P.

Coalgebraic representation of AJM-Games

- AJM-games can be represented as coalgebraic games via the tree of legal positions (plays), provided we perform a bisimilarity quotient on nodes.
- The class of coalgebraic games representing AJM-games is that of fixed polarized games.



Coalgebraic operations

Operations can be defined on coalgebraic games via (generalized) coiteration schemata.

Definition (Coiteration Schema)

$$\alpha_\Omega \circ h = Fh \circ \alpha_A$$

$$\begin{array}{ccc} A & \xrightarrow{h} & \Omega \\ \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_\Omega} \\ FA & \xrightarrow[Fh]{} & F\Omega \end{array}$$

Definition (Guarded Coiteration Schema)

$$\alpha_\Omega \circ h = F(g \circ Gh) \circ \delta_A$$

$$g : GA \to A$$

$$\begin{array}{ccc} A & \xrightarrow{h} & \Omega \\ \downarrow{\scriptstyle \delta_A} & & \downarrow{\scriptstyle \alpha_\Omega} \\ FGA & \xrightarrow[F(g \circ Gh)]{} & F\Omega \end{array}$$

[Cancila-Honsell-L., CMCS'2003]

Sum

- Sum is used to (de)compose games, [Conway01, Ch.14 “How to play several games at once in dozen of different way”].
- On mixed games, we define a sum, inspired by Conway disjunctive sum.
- On fixed games, we define a sum subsuming the tensor product of Game Semantics.
- The two sums have the same coalgebraic structure, but differ by the payoff on infinite plays.

Sum: coalgebraic structure and payoff

On $x + y$, at each step, the current player selects any of the components and makes a legal move in that component, the other remaining unchanged:

Definition

$$x + y = \{\langle a, x' + y\rangle \mid \langle a, x'\rangle \in x\} \uplus \{\langle a, x + y'\rangle \mid \langle a, y'\rangle \in y\}$$

$+ : (\mathbf{G}_{\mathcal{A}} \times \mathbf{G}_{\mathcal{A}}, \alpha_+) \longrightarrow (\mathbf{G}_{\mathcal{A}}, \mathrm{id})$ is a final morphism, for
$\alpha_+ : \mathbf{G}_{\mathcal{A}} \times \mathbf{G}_{\mathcal{A}} \longrightarrow F_{\mathcal{A}}(\mathbf{G}_{\mathcal{A}} \times \mathbf{G}_{\mathcal{A}})$ the coalgebra morphism:
$\alpha_+(x, y) = \{\langle a, \langle x', y\rangle\rangle \mid \langle a, x'\rangle \in x\} \uplus \{\langle a, \langle x, y'\rangle\rangle \mid \langle a, y'\rangle \in y\}$.

Payoff. Two sums arise from the above coalgebraic definition:

- Mixed sum $\oplus$. An infinite play is winning for L (R) if all infinite subplays in the components are winning for L (R), it is a draw otherwise.
- Fixed sum $\otimes$. An infinite play is winning for L iff all infinite subplays in the components are winning for L, it is winning for R otherwise.

Negation and linear implications

The roles of L/R are exchanged on $\overline{x}$.

Definition (Negation)

$$\overline{x} = \{\langle \overline{a}, \overline{x'} \rangle \mid \langle a, x' \rangle \in x\} \ .$$

$\overline{} : (\mathbf{G}_{\mathcal{A}}, \alpha_{-}) \longrightarrow (\mathbf{G}_{\mathcal{A}}, \mathrm{id})$ is a final morphism, for $\alpha_{-} : \mathbf{G}_{\mathcal{A}} \longrightarrow F_{\mathcal{A}}(\mathbf{G}_{\mathcal{A}})$ the coalgebra morphism:

$$\alpha_{-}(x) = \{\langle \overline{a}, x' \rangle \mid \langle a, x' \rangle \in x\}$$

The payoff on infinite plays of $\overline{x}$ is opposite to the payoff on x.

Definition (Linear Implications)

- On mixed games: $x \to y = \overline{x \oplus \overline{y}}$.
- On fixed games: $x \multimap y = \overline{x \otimes \overline{y}}$.

- same coalgebraic structure, different payoff
- $\overline{x \oplus y} = \overline{x} \oplus \overline{y}$, hence $x \to y = \overline{x} \oplus y$
- but $\overline{x \otimes y} \neq \overline{x} \otimes \overline{y}$

Infinite sum

The game $+^\infty x$ provides infinitely many copies of x.

Definition (Infinite Sum)

We define the infinite sum $+^\infty : \mathbf{G}_\mathcal{A} \longrightarrow \mathbf{G}_\mathcal{A}$ by:
$+^\infty x = \{\langle a, x' + (+^\infty x)\rangle \mid \langle a, x'\rangle \in x\}$.

The above is an instance of guarded coiteration:

- $$G(A) = \mathbf{G}_\mathcal{A} \times A$$
- the guard $g : G(\mathbf{G}_\mathcal{A}) \longrightarrow \mathbf{G}_\mathcal{A}$ is
$$g(x_1, x_2) = x_1 + x_2$$
- $\delta : \mathbf{G}_\mathcal{A} \to \mathcal{P}_{<\kappa}(\mathcal{A} \times \mathbf{G}_\mathcal{A} \times \mathbf{G}_\mathcal{A})$ is
$\delta_{\mathbf{G}_\mathcal{A}}(x) = \{(a_1, x_1', x) \mid (a_1, x_1') \in x\}$

Operations on polarized games

Definition (Pruning)

Let $(\ )_+, (\ )_- : \mathbf{G}_{\mathcal{A}} \longrightarrow \mathbf{G}_{\mathcal{A}}$ be the mutually recursive functions defined as:

$$\begin{cases} (x)_+ = \{\langle a, (x')_- \rangle \mid \langle a, x' \rangle \in x \ \& \ \lambda a = L\} \\ (x)_- = \{\langle a, (x')_+ \rangle \mid \langle a, x' \rangle \in x \ \& \ \lambda a = R\} \ . \end{cases}$$

We define the pruning operation as $(\ )_-$.

Definition (Polarized Operations)

Let x, y be polarized games. We define:

- the polarized sum as the game $(x + y)_-$;
- the polarized linear implication as the game $(x \multimap y)_-$;
- the polarized infinite sum as the game $(+^\infty x)_-$.

2. Categories of coalgebraic games

Joyal's paradigmatic categorical construction

Definition (The category $\mathcal{Y}$)

Objects: Conway's games

Morphisms: $\sigma : x \to y$ winning strategy for LII on $\overline{x} \oplus y$

Theorem

The category $\mathcal{Y}$ is compact closed with $\oplus$ as tensor.

Proposition

The category $\mathcal{Y}$ induces the following equivalence on games:

$$x \sim_{\mathcal{Y}} y \text{ iff } \exists \sigma : x \to y \text{ and } \exists \sigma' : y \to x\ .$$

$\sim_{\mathcal{Y}}$ is a congruence w.r.t. game operations.

Generalizing the above construction to mixed games and non-losing strategies is non-trivial. We proceed step by step.

The category $\mathcal{X}_{\mathcal{A}}$ of fixed games

Definition (The Category $\mathcal{X}_{\mathcal{A}}$)

Objects: fixed games.
Morphisms: $\sigma : x \to y$ winning strategy for LII on $x \multimap y$.

Identities on $\mathcal{X}_{\mathcal{A}}$ are the copy-cat strategies, and closure under composition is obtained via the swivel-chair (copy-cat) strategy.

Theorem

The category $\mathcal{X}_{\mathcal{A}}$ is $*$-autonomous, i.e. a model of the multiplicative fragment of Linear Logic.

Theorem

The category of polarized coalgebraic games and position independent winning strategies is equivalent to the category of AJM-games and winning strategies of [Abramsky96].

Extending the construction on mixed games

Extending the construction on mixed games is problematic, because non-losing strategies are not closed under composition.
One way out is considering partial strategies, see [Mellies2009, Categorical semantics of LL].
In [Honsell-L.-Redamalla2012, Equivalences and congruences on infinite Conway games] and [Honsell-L.-Redamalla2012, Categories of coalgebraic games], we pursue the approach of total strategies, in the line of Joyal's original construction.

The category $\mathcal{Y}_{\mathcal{A}}$ including mixed games

Idea: Mixed games x can be represented as pairs of fixed games $\langle x^-, x^+ \rangle$, where

- x^- is obtained from x by taking draw plays as winning for R
- x^+ is obtained from x by taking draw plays as winning for L.

Definition (The Category $\mathcal{Y}_{\mathcal{A}}$)

$$\mathcal{Y}_{\mathcal{A}} = \mathcal{X}_{\mathcal{A}} \times \mathcal{X}_{\mathcal{A}}$$

Objects: pairs of fixed games $x = \langle x_1, x_2 \rangle$.
Morphisms: pairs of winning strategies for LII $\langle \sigma_1, \sigma_2 \rangle$ on $\langle x_1 \multimap y_1, x_2 \multimap y_2 \rangle$.

Theorem

The category $\mathcal{Y}_{\mathcal{A}}$ is $*$-autonomous.

Theorem

The full subcategory of $\mathcal{Y}_{\mathcal{A}}$, consisting of well-founded games and winning strategies, is compact closed.

Corollary (Joyal77)

The category of Conway games and winning strategies is compact closed.

Categorical equivalence

Definition

$\mathcal{Y}_{\mathcal{A}}$ induces a pre-equivalence on mixed games $x = \langle x^-, x^+ \rangle, y = \langle y^-, y^+ \rangle$:

$$x \leq_{\mathcal{Y}_{\mathcal{A}}} y \text{ iff}$$

$\exists$ a pair of winning strategies for LII on $\overline{x^- \otimes \overline{y^-}}$ and $\overline{x^+ \otimes \overline{y^+}}$.

Definition (Loopy pre-equivalence, Berlekamp-Conway-Guy82)

For x, y loopy games, we define:

$$x \leq_l y \text{ iff}$$

$\exists$ a pair of non-losing strategies for LII on $\overline{x^- \oplus \overline{y^-}}$ and $\overline{x^+ \oplus \overline{y^+}}$.

Theorem (Categorical Characterization)

For loopy games x, y,

$$x \leq_l y \iff x \leq_{\mathcal{Y}_2} y .$$

3. From sequential to concurrent games

Selective sum

In the disjoint sum, at each step, the current player moves in exactly one component (interleaving semantics), while in the selective sum, the current player (or a team of players) can possibly move in both components, *i.e.* in different parts of the board simultaneously (parallel semantics).

- Selective sum on coalgebraic games: $x \vee y = \{\langle a, x' \vee y \rangle \mid \langle a, x' \rangle \in x\} \uplus \{\langle b, x \vee y' \rangle \mid \langle b, y' \rangle \in y\} \uplus \{\langle \langle a, b \rangle, x' \vee y' \rangle \mid \langle a, x' \rangle \in x \;\&\; \langle b, y' \rangle \in y \;\&\; \lambda a = \lambda b\}$.
- Negation and linear implication:

$$\overline{x} = \{\langle \overline{a}, \overline{x'} \rangle \mid \langle a, x' \rangle \in x\} \qquad x \multimap y = \overline{x} \vee y \; .$$

Applying Joyal's construction on coalgebraic games with selective sum

Coalgebraic games and winning strategies with selective sum fail to form a category even when only finite games or partial strategies are considered.

However:

Theorem

Polarized games and partial strategies form a symmetric monoidal closed category where tensor is selective sum.

The copy-cat strategies are the identities.
Composition is obtained via the copy-cat strategy, and a non-standard parallel application of strategies.

Categories of fixed/mixed polarized games with total strategies can be also built, see [Honsell-L.-Pellarini2014, Categories of Coalgebraic Games with Selective Sum].

Question: what is the status of this category?

The category of coalgebraic games with selective sum turns out to be equivalent to a new category of multigames.

Beyond sequential games ... towards concurrent games.

Sequential vs concurrent games

- Sequential games:
  - at each step, either L or R player moves (global polarization),
  - only a single move can be performed at each step.
- In the context of Game Semantics, concurrent games have been introduced [Abramsky-Mellies99, Ghica-Menaa10,Winskel et al.11-,. . .], where
  - global polarization is abandoned,
  - multiple moves are allowed.

Coalgebraic multigames

Coalgebraic multigames are situated half-way between traditional sequential games and concurrent games:

- global polarization is still present,
- however multiple moves are possible at each step, *i.e.* a team of L/R players moves in parallel.

Definition (Coalgebraic Multigames)

- Let $\mathcal{M}_{\mathcal{A}}$ be the set of multimoves, *i.e.* the powerset of all finite sets of atomic moves with the same polarity,

$$\mathcal{M}_{\mathcal{A}} = \{\alpha \in \mathcal{P}_f(\mathcal{A}) \mid \forall a, a' \in \alpha.\ \lambda a = \lambda a'\} \ .$$

- Let $F_{\mathcal{M}_{\mathcal{A}}} : \mathrm{Set}^* \to \mathrm{Set}^*$ be the functor defined by

$$F_{\mathcal{M}_{\mathcal{A}}}(A) = \mathcal{P}_{<\kappa}(\mathcal{M}_{\mathcal{A}} \times A) \ .$$

- Let $(\mathbf{M}_{\mathcal{A}}, id)$ be the final $F_{\mathcal{M}_{\mathcal{A}}}$-coalgebra.

A coalgebraic multigame is an element X of the carrier $\mathbf{M}_{\mathcal{A}}$ of the final coalgebra.

Multigame sum

On the multigame sum, at each step, the next player selects either one or both components, and makes a move in the selected components, while the component which has not been chosen (if any) remains unchanged.

Definition (Sum)

The sum of two multigames $\triangledown : \mathbf{M}_{\mathcal{A}} \times \mathbf{M}_{\mathcal{A}} \longrightarrow \mathbf{M}_{\mathcal{A}}$ is defined by:

$$\begin{aligned} X \triangledown Y = & \{\langle \alpha, X' \triangledown Y \rangle \mid \langle \alpha, X' \rangle \in X\} \uplus \\ & \{\langle \beta, X \triangledown Y' \rangle \mid \langle \beta, Y' \rangle \in Y\} \uplus \\ & \{\langle \alpha \uplus \beta, X' \triangledown Y' \rangle \mid \langle \alpha, X' \rangle \in X \,\&\, \langle \beta, Y' \rangle \in Y\} \,. \end{aligned}$$

Negation, linear implication

Definition (Negation)

The negation $\overline{}$: $\mathbf{M}_{\mathcal{A}} \longrightarrow \mathbf{M}_{\mathcal{A}}$ is defined by:

$$\overline{X} = \{\langle \overline{\alpha}, \overline{X'} \rangle \mid \langle \alpha, X' \rangle \in X\} \; .$$

Definition (Linear Implication)

The linear implication of the multigames X, Y, $X \multimap Y$, is defined by

$$X \multimap Y = \overline{X} \triangledown Y \; .$$

A monoidal closed category of multigames and strategies

Let $\mathcal{Y}_{\mathbf{M}_{\mathcal{A}}}$ be the category defined by:
Objects: polarized multigames.
Morphisms: $\sigma : X \multimap Y$ partial strategy for LII on $X \multimap Y$.
Identity: the identity $id_X : X \multimap X$ is the copy-cat strategy. On the polarized multigame $X \multimap X = (\overline{X} \triangledown X)$, R can only open on X, then L proceeds by copying the move on $\overline{X}$ and so on, at each step R has exactly one component to move in.

Theorem

The category $\mathcal{Y}_{\mathbf{M}_{\mathcal{A}}}$ is symmetric monoidal closed, and it is equivalent to the category of polarized coalgebraic games with selective sum.

Question: What are multigames useful for? Which level of parallelism do they capture?

They provide a model of PCF with parallel OR, see [Honsell-L.2014, Polarized Multigames]

Conclusions and future work

- Coalgebraic games
  - provide a unified (simplified) treatment of games
  - subsume (non-wellfounded) Conway games, Game Semantics, games in Economics
  - shed light on the relations between them.
- Coalgebraic multigames
  - are situated halfway between traditional sequential and recent concurrent games
  - help in clarifying the steps bringing from sequential to concurrent games

? Investigating usefulness of the selective model for modeling distributed systems.

? Extending the selective model by abandoning global polarization, towards concurrent coalgebraic games.

? Generalizing the coalgebraic framework in particular towards imperfect information games.